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THE TRANSITION CURVE.

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THE TRANSITION CURVE

OR

CURVE OF ADJUSTMENT

BY THE METHOD OF RECTANGULAR CO-ORDINATES AND BY DEFLECTION ANGLES

(POLAR CO-ORDINATES)

BASED ON THE FRENCH OF
M. NORDLING

WITH ADDITIONAL PROBLEMS

BY

N. B. KELLOGG
M. AM. SOC. C. E.

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PREFACE.

The chapter devoted to the cubic parabola is derived from the method introduced by M. Nordling, as explained by him in the "Annales des Ponts et Chaussées" 1867. Translations of a portion of this article have appeared from time to time, but only so far as related to connecting a straight line with a circular curve. That portion relating to connecting circular curves of different radii, by means of the cubic parabola, has not appeared in the form given by Nordling, so far as I am aware. The formulæ deduced in the latter case are of general application and equally true for connecting curve with curve or curve with tangent, when proper values are introduced into the equations.

That of joining a tangent with a curve is a special case where one of the radii becomes infinitely great. Some of the recent spirals, adopted as curves of adjustment in railroad location, easily develop from the equations of the cubic parabola by making the proper substitutions in them.

Following the supposition indicated by M. Nordling i. e., considering $x = L$ and substituting L for x in the equations of Chapter I, the formulæ of (61—70), were obtained by the writer during the summer of 1884.

While the theory of the transition curve as here developed is based upon that of Froude or Nordling, it is believed that some of the problems and tables, used in its application, have not appeared before.

The figures used to illustrate the transition curve represent spirals, as no interest attaches to the cubic parabola in this connection after it reaches its minimum radius, which occurs when its central angle becomes 24°06′; up to that point the figures serve as well for the one as the other. OQ represents the relative position of the cubic parabola to that of the spiral having the same origin. The examples given for illustration throughout Chapters I and II, are based upon the same data, so that the results by the different methods, may be readily compared; within moderate limits it will be observed the differences are small. A few simple equations of the calculus are used to derive the formulæ, but a knowledge of it is not necessary for their application.

The larger type may be read independently of the smaller type. The latter may be used when the curve is extended beyond ordinary limits, and greater accuracy in results is sought.

By the "length of the inclined plane" (really a warped surface) the "run off" is to be understood, *i. e.* the length of the transition curve.

The more important formulæ are in full faced type.

In the arrangement of the text, I am indebted to Prof. H. I. Randall for valuable suggestions.

N. B. K.

San Francisco, 1899.

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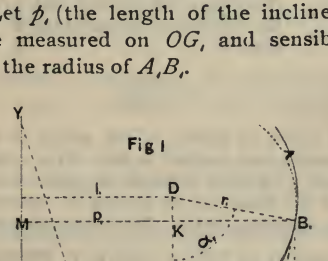
CHAPTER I.

THEORY OF THE CUBIC PARABOLA, OR TRANSITION CURVE, AS APPLIED TO THE ALIGNMENT OF RAILROADS.

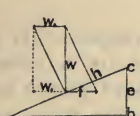
§ 1. Let OG, and A_1B_1, (Fig. 1) be respectively, a tangent and circular curve to be united by a cubic parabola. Let p, (the length of the inclined plane) be the distance measured on OG, and sensibly equal to $OB_1 = L_1$; r, the radius of A_1B_1.



From mechanics, the expression for the centrifugal force of a body moving in a curvilinear path is:

$*f = \frac{wv^2}{32.2r}$ from which we determine the superelevation of the outer rail to produce a horizontal force due to gravity

to neutralize this centrifugal force, this superelevation is: $e = \frac{gv^2}{32\ 2r}$ in which r_1 = the radius of curvature of the path in feet; v = the velocity of the moving body in feet per second, and g = the gauge of the track, or width of the path in feet. 32.2 = acceleration of gravity in feet per second. The transverse inclination of the track (or "cant") to produce the horizontal force due to gravity, must be acquired gradually, and if we take a distance "i" to rise one unit in height, then $\frac{1}{i}$ will denote the



*Let w = the weight of a moving body constrained to move in a curvilinear path f = the centrifugal force developed by the weight revolving about a center with a radius r. Then the expression by mechanics for the centrifugal force is

$$f = \frac{wv^2}{32.2r}; \qquad (1)$$

Conceive the center of gravity of the weight to be moved in a direction contrary and parallel to that of the centrifugal force—while its center of support remains the same—until a horizontal component of the force due to gravity is developed equal and opposed to the centrifugal force. Denote this component by w_1; then by similar triangles we have $w : w_1 = g : e$; whence $w_1 = \frac{we}{g}$ now by the conditions imposed

$$w_1 = f \therefore \frac{we}{g} = \frac{wv^2}{32.2r} \text{ or } \frac{e}{g} = \frac{v^2}{32.2r} \text{ or}$$

$$e = \frac{gv^2}{32.2r};$$

The last formulæ is a close approximation and the one generally used. It is defective in considering g a constant while it is really a decreasing function of e and is used in place of ac which is the constant. The error, however, is small in all cases where it is used for superelevation. It may not be out of place to remark here, that it is the usual custom in fixing superelevation, to depress the inner rail below the grade of the center line of the track ½e and elevate

the outer rail above the grade line $\frac{1}{2}e$, thus preserving the center line at grade.

A more nearly correct value for e is

$$e = \frac{hv^2}{\sqrt{(32.2)^2 r^2 + v^4}}.$$

The resultant of w and w_1 due to the effect of gravity $= \sqrt{w^2 + w_1^2}$ which is the pressure normal to the plane of the path when centrifugal force is developed.

rise in a unit of distance and for any distance p the rise will be $\frac{p}{i}$ = the superelevation necessary to produce a horizontal force due to gravity, equal and opposed to the centrifugal force, hence:

$$e = \frac{p_1}{i} = \frac{gv^2}{32.2\, r_1} \text{ whence } r_1 p_1 = \frac{gv^2 i}{32.2} \qquad (2)$$

The second member of the equation, for assumed values of v and i, is constant, and is placed equal to P, hence:

$$r_1 p_1 = P; \qquad r_1 = \frac{P}{p_1}. \qquad (3)$$

§ 2. From the calculus the expression for the radius of curvature is; $r_1 = \frac{ds^3}{dp_1\, d^2 y_1}$, in which ds is the differential of OB_1 and dp_1 the differential of OG_1 (or SB_1). Fig. 2; d^2y the second differential of B_1G_1 with respect to p_1; now as we have assumed OG_1 sensibly equal to OB_1, therefore we may assume $ds = dp_1$, which gives, thereby:

$$r_1 = \frac{dp_1^3}{dp_1\, d^2 y_1} = \frac{P}{p_1} \text{ inverting the terms } \frac{d^2 y_1}{dp_1^2} = \frac{p_1}{P}; \qquad (4)$$

whence

$$\frac{d^2 y_1}{dp_1} = \frac{p_1\, dp_1}{P}, \text{ integrating, } \frac{dy_1}{dp_1} = \frac{p_1^2}{2P}; \qquad (5)$$

whence :

$$dy_{,} = \frac{p_{,}^{2} dp_{,}}{2P}, \text{ integrating, } y_{,} = \frac{p_{,}^{3}}{6P}; \qquad (6)$$

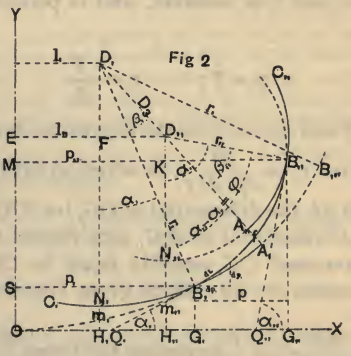
in which $y_{,}$ and $p_{,}$ are co-ordinates of the point $B_{,}$. By the same process we may find for the point $B_{,,}$ of the parabola at which the radius of the circle is $r_{,,}$, being also the radius of the cubic parabola at the same point and coincident with it,

$$y_{,,} = \frac{p_{,,}^{3}}{6P}, \text{ which is the equation of a cubic parabola.} \qquad (7)$$

In general for any point of the cubic parabola whose co-ordinates are x and y we will have :

$$y = \frac{x^3}{6P}. \qquad (8)$$

EXAMPLE 1. Given $x = 150$; $P = 171900$ to find the ordinate y by (8) $y = \frac{x^3}{6P} = \frac{(150)^3}{6 \times 171900} = \frac{3375000}{1031400} = 3.27$.



§ 3. From the relation established in equation (3), we have :

$$p_{,,} = \frac{P}{r_{,,}}; \quad p_{,} = \frac{P}{r_{,}}. \qquad (9)$$

If now we denote in Fig. 2

$$p = p_{,,} - p_{,} = \frac{P}{r_{,,}} - \frac{P}{r_{,}} = P\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right)$$
$$= P\,\frac{(r_{,} - r_{,,})}{r_{,,}r_{,}} = G_{,}G_{,,}; \qquad (10)$$

then

$$P = \frac{p r_{,} r_{,,}}{r_{,} - r_{,,}}; \qquad (11)$$

which substituted in the values for $p_{,}$ and $p_{,,}$ give

$$p_{,} = \frac{p r_{,,} r_{,}}{r_{,}(r_{,} - r_{,,})} = \frac{p r_{,,}}{r_{,} - r_{,,}}; \quad p_{,,} = \frac{p r_{,} r_{,,}}{r_{,,}(r_{,} - r_{,,})} = \frac{p r_{,}}{r_{,} - r_{,,}}. \qquad (12)$$

By(5) $$\frac{dy_{,}}{dp_{,}} = \tan \alpha_{,} = \frac{p_{,}^{2}}{2P}; \quad \frac{dy_{,,}}{dp_{,,}} = \tan \alpha_{,,} = \frac{p_{,,}^{2}}{2P}; \qquad (13)$$

and since the angles are small, the arcs may be taken for the tangents of the angles, thus:

$$\frac{dy_{,}}{dp_{,}} = \alpha_{,} = \frac{p_{,}^{2}}{2P}; \qquad \frac{dy_{,,}}{dp_{,,}} = \alpha_{,,} = \frac{p_{,,}^{2}}{2P}; \qquad (14)$$

EXAMPLE 2. Given $x_{,,} = p_{,,} = 150$; $P = 171900$ to find the angle $\alpha_{,,}$ subtended by $L_{,,}$: $\tan \alpha_{,,} = \frac{p_{,,}^{2}}{2P}$ (supposed = arc $\alpha_{,,}$)

$$\alpha_{,,} = \frac{(150)^{2}}{2 \times 171900} = \frac{22500}{343800} = .065 = \alpha_{,,}{}^{\circ} = 3^{\circ}\ 43'+$$

Substituting the values of $p_{,}$ and P in terms of p, $r_{,}$ and $r_{,,}$ we have:

$$\alpha_{,} = \frac{p r_{,,}}{2 r_{,}(r_{,} - r_{,,})}; \qquad \alpha_{,,} = \frac{p r_{,}}{2 r_{,,}(r_{,} - r_{,,})}; \qquad (15)$$

subtracting

$$\alpha_{,,} - \alpha_{,} = \frac{p\,(r_{,}^{2} - r_{,,}^{2})}{2 r_{,} r_{,,}(r_{,} - r_{,,})} = \frac{p\,(r_{,} + r_{,,})(r_{,} - r_{,,})}{2 r_{,} r_{,,}(r_{,} - r_{,,})}; \qquad (16)$$

whence

$$* \quad \alpha_{,,} - \alpha_{,} = \frac{p}{2}\,\frac{(r_{,} + r_{,,})}{r_{,} r_{,,}} = \frac{p}{2}\left(\frac{1}{r_{,,}} + \frac{1}{r_{,}}\right) = \beta_{,,} + \beta_{,}; \qquad (17)$$

EXAMPLE 3. Given $x = p = 150$; $r_{,} = 2865$; $r_{,,} = 818.8$, to find by (17):

$$\alpha_{,,} - \alpha_{,} = \beta_{,,} + \beta_{,} = \phi = \frac{p}{2}\left(\frac{1}{r_{,}} + \frac{1}{r_{,}}\right) = 75\,(.00122 + .00035) = 75 \times .00157$$

$$\alpha_{,,} - \alpha_{,} = \beta_{,,} + \beta_{,} = \phi = \text{arc } .11775 = 6^{\circ}\,44' +$$

$\beta_{,} = A_{,}D_{,}B_{,}$; $\beta_{,,} = A_{,,}D_{,,}B_{,,}$; if now we make alternately $r_{,}$ and $r_{,,} = \infty$ and, at the same time, make $\beta_{,} = o$ and $\beta_{,,} = o$, we have:

$$\beta_{,} = \frac{p}{2r_{,}};\ \beta_{,,} = \frac{p}{2r};\ \beta_{,}r_{,} = \tfrac{1}{2}p;\ \beta_{,,}r_{,,} = \tfrac{1}{2}p; \qquad (18)$$

whence

$$\beta_{,,}r_{,,} = \beta_{,}r_{,} = \tfrac{1}{2}p;\ \beta_{,}r_{,} + \beta_{,,}r_{,,} = p; \qquad (19)$$

*The values of $\beta_{,}$ $\beta_{,,}$ $\alpha_{,}$ $\alpha_{,,}$ and ϕ are expressed in arc at radius = 1. The arc of a circle which is equal to the radius in length = 57.295. The arc of 1 degree at radius = 1 is .01745 +. Hence the expression for any number of degrees in a given circular arc is $A^{\circ} = \frac{\text{arc } A}{.01745+}$.

§ 4. **To find the distance $A_{,}A_{,,} = f$ between the two circular curves measured on a common normal passing through their centers; i. e., the least distance apart of the circular curves to be joined by the cubic parabola:** Referring the co-ordinates of the circle with radius $r_{,}$ to the same origin as the cubic parabola in which $m_{,} = N_{,}H_{,}$, we have:

$$r_{,}^{2} = (p_{,} - l_{,})^{2} + (y_{,} - (r_{,} + m_{,}))^{2}; \qquad (20)$$

squaring the last term of the second member then,

$$r_{,}^{2} = (p_{,} - l_{,})^{2} + y_{,}^{2} - 2r_{,}y_{,} - 2y_{,}m_{,} + 2r_{,}m_{,} + m_{,}^{2} + r_{,}^{2}; \qquad (21)$$

suppressing all terms of $y_{,}$ and $m_{,}$ in which $r_{,}$ does not enter as a factor, since $y_{,}$ and $m_{,}$ are small compared with

$r_{,,}$ (the opposition of signs will still further diminish the error) and factoring the remaining terms, we have:

$$2r_{,}(y_{,}-m_{,}) = (p_{,}-l_{,})^2; \qquad (22)$$

in which $l_{,}$ is the abscissa of the center of the circle and $= \frac{1}{2}p_{,}$

$$y_{,} = m_{,} + \frac{(p_{,}-l_{,})^2}{2r_{,}} = m_{,} + \frac{p_{,}^2}{8r_{,}} = \frac{p_{,}^2}{6r_{,}}; \qquad (23)$$

whence

$$m_{,} = \frac{p_{,}^2}{6r_{,}} - \frac{p_{,}^2}{8r_{,}} = \frac{2p_{,}^2}{48r_{,}} = \frac{p_{,}^2}{24r_{,}}, \text{ or in general, } \mathrm{m} = \frac{\mathrm{x}^2}{\mathbf{24r}}; \qquad (24)$$

EXAMPLE 4. Given $x = 150$; $r = 1146$ to find the length of the normal "m" common to and between the tangent and circular curve to be joined by a cubic parabola: By the general equation

$$m = \frac{x^2}{24r} = \frac{(150)^2}{24 \times 1146} = \frac{22500}{27504} = 0.81.$$

In the same way we may show that

$$m_{,,} = \frac{p_{,,}^2}{24r_{,,}} \text{ in which } m_{,,} = N_{,,}H_{,,} \qquad (25)$$

To prove that the cubic parabola and circle have a common tangent at $p_{,}y_{,}$; i. e., are tangent to each other, we differentiate their respective equations: for the circle,

$$\frac{dy_{,}}{dp_{,}} = \frac{p_{,}-l_{,}}{r_{,}} = \frac{p_{,}}{2r_{,}} \text{ (since } l_{,} = \tfrac{1}{2}p_{,}) = \tan \alpha_{,}; \qquad (26)$$

for the cubic parabola,

$$\frac{dy_{,}}{dp_{,}} = \frac{p_{,}^2}{2P} = \frac{p_{,}}{2r_{,}} \text{ (since } P = r_{,}p_{,}) = \tan \alpha_{,}; \qquad (27)$$

hence the tangents coincide.

Substituting the value of $p_{,}$ in terms of p

$$m_{,} = \frac{p^2 r_{,,}^2}{24r_{,}(r_{,}-r_{,,})^2} \quad \text{also} \quad m_{,,} = \frac{p^2 r_{,}^2}{24r_{,,}(r_{,}-r_{,,})^2}; \qquad (28)$$

we now have $m_{,}$ and $m_{,,}$ in known terms, and by placing their values in the equations for $y_{,}$ and $y_{,,}$ in which the

ordinates for the parabola and circles of radii r_i and r_{ii} have the same abscissa x we have for the parabola,

$$y = \frac{x^3}{6P}; \tag{29}$$

for the circles $y_i = m_i + \frac{(x-l)^2}{2r_i}$; $y_{ii} = m_{ii} + \frac{(x-l_{ii})^2}{2r_{ii}}$;

substituting in equation for y, $P = \frac{pr_i r_{ii}}{r_i - r_{ii}}$;

$$y = \frac{(r_i - r_{ii})\, x^3}{6\, pr_i r_{ii}};$$

$$y_i = \frac{pr_{ii}^2}{24r_i(r_i - r_{ii})^2} + \frac{(2(r_i - r_{ii})x - pr_{ii})^2}{8r_i\,(r_i - r_{ii})} = \frac{p^2 r_{ii}^2 + 3\,[2(r_i - r_{ii})\,x - pr_{ii}]^2}{24r_i(r_i - r_{ii})^2} \tag{30}$$

by the same process

$$y_{ii} = \frac{p^2 r_i^2 + 3\,[2\,(r_i - r_{ii})\,x - pr_i]^2}{24r_{ii}\,(r_i - r_{ii})^2}; \tag{31}$$

$$y - y_i = \frac{(r_i - r_{ii})\,x^3}{6pr_i r_i} - \frac{3\,[2\,(r_i - r_{ii})\,x - pr_{ii}]^2 + p^2 r_{ii}^2}{24r_i\,(r_i - r_{ii})}; \tag{32}$$

$$y_{ii} - y = \frac{-(r_i - r_{ii})\,x^3}{6pr_i r_{ii}} + \frac{3\,[2\,(r_i - r_{ii})x - pr_i]^2 + p^2 r_i^2}{24r_{ii}\,(r_i - r_{ii})}; \tag{33}$$

If now we substitute for x; $p_i + \frac{1}{2}p = p_{ii} - \frac{1}{2}p$ and $P = \frac{pr_i r_{ii}}{r_i - r_{ii}}$ in equations for the values of y, y_i and y_{ii},

$$y = \frac{x^3}{6P} = \frac{p^3(r_i + r_{ii})^3}{8(r_i - r_{ii})^3}; \left(\frac{r_i - r_{ii}}{6pr_i r_{ii}}\right) = \frac{p^2(r_i + r_{ii})^2}{48r_i r_{ii}(r_i - r_{ii})^2}; \tag{34}$$

$$y_i = \frac{p^2 r_{ii}^2 + 3\left[2(r_i - r_{ii})\,\frac{p}{2}\,\frac{r_i + r_{ii}}{r_i - r_{ii}} - pr_i\right]^2}{24r_i(r_i - r_{ii})^2}; \tag{35}$$

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$$y_{,}=\frac{p^2 r_{,,}{}^2+3\,[pr_{,}+pr_{,,}-pr_{,,}]^2}{24r_{,}(r_{,}-r_{,,})^2}=\frac{p^2(r_{,,}{}^2+3r_{,}{}^2)}{24r_{,}(r_{,}-r_{,,})^2};\quad(36)$$

multiply and divide the second member of the equation by $2r_{,,}$

$$y_{,}=\frac{p^2(2r_{,,}{}^3+6r_{,}{}^2r_{,,})}{48r_{,}r_{,,}(r_{,}-r_{,,})^2};\quad(37)$$

$$y-y_{,}=\frac{p^2(r_{,}+r_{,,})^3-p^2(2r_{,,}{}^3+6r_{,}{}^2r_{,,})}{48r_{,}r_{,,}(r_{,}-r_{,,})^2}=$$
$$\frac{p^2[(r_{,}+r_{,,})^3-2r_{,,}{}^3-6r_{,}{}^2r_{,,}]}{48r_{,}r_{,,}(r_{,}-r_{,,})^2}\quad(38)$$

$(r_{,}+r_{,,})^3=r_{,}{}^3+3r_{,}{}^2r_{,,}+3r_{,}r_{,,}{}^2+r_{,,}{}^3$; substituting, cancelling and factoring,

$$y-y_{,}=\frac{p^2(r_{,}-r_{,,})^3}{48r_{,}r_{,,}(r_{,}-r_{,,})^2}=\frac{p^2(r_{,}-r_{,,})}{48r_{,}r_{,,}};\quad(39)$$

By similar reduction,

$$y_{,,}-y=\frac{p^2(r_{,}-r_{,,})}{48r_{,}r_{,,}};\quad(40)$$

which is exactly the same value obtained for $y-y_{,,}$ which shows that the cubic parabola bisects f (or m) and each expression equals $\frac{1}{2}f$ whence adding the values $y_{,,}-y+y-y_{,}=y_{,,}-y_{,}$;

$$y_{,,}-y_{,}=\frac{p^2(r_{,}-r_{,,})}{24r_{,}r_{,,}}=\frac{p^2}{24}\left(\frac{1}{r_{,,}}-\frac{1}{r_{,}}\right)=f=A_{,}A_{,,};\quad(41)$$

if the central angle of the cubic parabola be small, not exceeding say 10°.

EXAMPLE 5. Given $x=p=150$, $r_{,,}=818.8$, $r_{,}=2865$ to find the offset by (41),

$$f=\frac{p^2}{24}\left(\frac{1}{r_{,,}}-\frac{1}{r_{,}}\right)=A_{,}A_{,,}=\frac{(150)^2}{24}(.00122-.00035)=$$
$$\frac{22500\times.00087}{24}=0.815.$$

§ 5. From the general equation $y = \frac{x^3}{6P}$ we see that the ordinates vary as the cube of their abscissas. Having computed the value of any ordinate y, we have for any other ordinate y_o;

$$y : y_o = \frac{x^3}{6P} : \frac{x_o^3}{6P}; \qquad (42)$$

or

$$y_o = \frac{\frac{yx_o^3}{6P}}{\frac{x^3}{6P}} = \frac{yx_o^3}{x^3}, \text{ hence } y_o = \left(\frac{x_o}{x}\right)^3 y. \qquad (43)$$

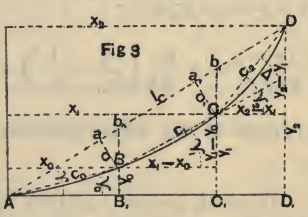
EXAMPLE 6. Given $x = 120$, $x_o = 150$, $y = 1.67$ to find any other ordinate y_o by (42),

$$y_o = \left(\frac{x_o}{x}\right)^3 y = (1.25)^3 \times 1.67 = 3.27.$$

This will facilitate the computation of any ordinate following the first.

It is to be noted that all of the above equations are approximately true only for small central angles.

§ 6. To determine the method whereby the cubic parabola may be laid out by deflection angles, and the length of the corresponding chords,



we will assume equal increments of x of finite length, (though they may be taken unequal) from which the corresponding chords (of unequal length) may be computed as well as the deflection angles.

By a previous equation

$$\frac{dy}{dx}=\frac{x^2}{2P}=\frac{x^3}{2Px}=\frac{3x^3}{6Px}=\frac{3y}{x}=\frac{y}{\frac{1}{3}x}=\tan\alpha. \qquad (44)$$

EXAMPLE 7. Given $x=150$; $y=3.26$, to find α by (44);

$$\tan\alpha=\frac{y}{\frac{1}{3}x}=\frac{3.26}{50}=.0652=\tan 3^\circ 44'$$

which shows that the tangent of the angle which a tangent to the curve makes with the axis of x equals the ordinate divided by $\frac{1}{3}$ the abscissa of the same point. If, therefore, we have as in Fig. 3, the cubic parabola $ABCD$,

$$\frac{y_o}{x_o}=\tan BAB_1; \qquad (45)$$

and

$$\frac{3y_o}{x_o}=\tan\alpha_o; \qquad (46)$$

then

$$\tan BAB_1=\frac{\tan\alpha}{3}=\tan\gamma_o; \qquad (47)$$

$$\tan\gamma_1=\frac{y_1-y_o}{x_1-x_o}; \qquad (48)$$

$$\tan\gamma_2=\frac{y_2-y_1}{x_2-x_1}; \qquad (49)$$

$\gamma_o\gamma_1\gamma_2$ are the angles the respective chords make with the axis of X: $\gamma_2-\gamma_1$ is the angle which chord CD makes with BC. By a similar process the angle which chord AC makes with CD can be computed. For small central angles the tangents of the angles may be assumed equal the arcs so that

$$\frac{y_2-y_1}{x_2-x_1}-\frac{y_1-y_o}{x_1-x_o}=\gamma_2-\gamma_1=\triangle \qquad (50)$$

the exterior angles between the chords.

The several angles can be computed and tabulated, to any number which is likely to be needed, to conform to any system of "change points" determined upon after x_o y_o &c. have been computed for the particular transition curve where value of P has been fixed in conformity with the character of the alignment. The foregoing method for computing deflection angles is equally applicable to the spiral.

EXAMPLE 8. Given $x_o = 60$; $x_1 = 120$; $x_2 = 209.65$; $y_o = .21$; $y_1 = 1\ 67$; $y_2 = 8.91$, to find the deflection angles for laying out a cubic parabola which successive chords make with each other

$$\frac{y_o}{x_o} = \tan \gamma_o = \frac{.21}{60} = .0035 = \tan 0° 12';$$

$$\frac{y_1 - y_0}{x_1 - x_0} = \tan \gamma_1 = \frac{1\ 46}{60} = .0245 = \tan 1° 24';$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \tan \gamma_2 = \frac{7\ 24}{89.65} = \tan 4° 37';$$

$$\gamma_1 - \gamma_0 = 1° 21' - 0° 12' = 1° 12';$$

$$\gamma_2 - \gamma_1 = 4° 37' - 1° 24' = 3° 13';$$

For long chord; $\frac{y_2}{x_2} = \tan \gamma = \frac{8\ 91}{209\ 65} = 2° 27';$

comparing the value of α_{11} and γ we see $\gamma = \frac{\text{long } \alpha_{11}}{3}$.

§ 7. **To determine the ordinates** o, o_1 &c. From any chord as Ic, let $aB = o$, $B_1b = s$, $AB_1 = x_o$, $B_1B = y_o$, $bAB_1\ \gamma$. Then from Fig. 3, $\frac{s}{x_o} = \tan \gamma$, $s = x_o \tan \gamma$, $s - y_o = x_o \tan \gamma - y_o$, $\frac{o}{s - y_o} = \cos \gamma$, $o = (s - y_o) \cos \gamma$, or since $s = x_o \tan \gamma$,

$$o = (x_o \tan \gamma - y_o) \cos \gamma = x_o \sin \gamma - y_o \cos \gamma; \quad (51)$$

For the distances Aa &c., $Aa = A_1B_1$, $\sec \gamma - aB$, $\tan \gamma = x_o$, $\sec \gamma - o \tan \gamma$. In the same way we may obtain o_1 and Aa_1 to determine the point B; C &c. by measurement alone: First compute and lay off

the distances Aa, Aa_1 &c., then lay off AB and aB simultaneously; next BC and a_1C &c.; when γ is small the distances bB and b_1C may be used, distances Ab and Ab_1 being computed and laid off first.

§ 8. As a check it will be an advantage to compute the length of the long cord as well as the angle it makes with the axis of x, thus:

$$\tan \mathrm{DAD}_1 = \frac{\mathrm{DD}_1}{\mathrm{AD}_1} = \frac{y_n}{x_n}, \qquad (52)$$

in which (n) equals the number of increments or stations between A and D. Let the length of any chord be c_1, c_2 or c_3; then

$$c_2 = (x_2 - x_1) \sec \gamma_2; \quad c_4 = (x_4 - x_3) \sec \gamma_4; \quad (53)$$

EXAMPLE 9. Given $x_2 = 209.65$; $x_1 = 120$; $\gamma_2 = 4° 37'$, to find the length of chord, $c_2 = (x_2 - x_1) \sec \gamma_2 = 89.65 \times 1.0032 = 89.93$. For the length of long chord (from origin), $x_2 = 209.65$; $\gamma = 2° 27'$; $x_2 \sec \gamma = 209.65 \times 1.009 = 209.84$.

If the central angle is considerable, the error arising from considering the arcs equal the tangents of the angle, will be too great, and the expression tan α, tan γ &c. must be adhered to. It will be observed that the field work will often be facilitated by laying out the parabola by ordinates and calculating its length to preserve correct measurements along the line.

§ 9. **For the rectification of curves:** by the calculus we have

$$dz^2 = dx^2 + dy^2 = dx^2 + \frac{dy^2}{dx^2} dx^2, \qquad (54)$$

and since

$$\frac{dy}{dx} = \frac{x^2}{2P}; \quad \frac{dy^2}{dx^2} = \frac{x_4}{4P^2}; \qquad (55)$$

substituting and factoring

$$dz^2 = \left(1 + \frac{x^4}{4P}\right) dx^2 \therefore dz = \sqrt{1 + \frac{x^4}{4P^2}}\, dx; \quad (56)$$

developing the radical by the binomial formulæ

$$dz = \left[1 + \frac{x^4}{8P^2} - \frac{x^8}{128P^4} \cdots \frac{x^n}{NP^{\frac{n}{2}}}\right] dx; \quad (57)$$

Integrating the last equations between the limits $L_{,,}$ and $L_{,} = L$, $x_{,,}$ and $x_{,}$ we have

$$L = x_{,,}\left(1 + \frac{x_{,,}^4}{40P^2} - \frac{x_{,,}^8}{1152P^4} + \right) - x_{,}\left(1 + \frac{x_{,}^4}{40P^2} - \frac{x_{,}^8}{1152P^4} + \right) \quad (58)$$

since $P = r_{,,} x_{,,} = r_{,} x_{,}$

$$L = x_{,,}\left(1 + \frac{x_{,,}^2}{40r_{,,}^2} - \&c.\right) - x_{,}\left(1 + \frac{x_{,}^2}{40r_{,}^2} - \&c.\right); \quad (59)$$

in which L = the length of the cubic parabola uniting the curves with radii $r_{,}$ and $r_{,,}$. If $r_{,} = \infty$, $L_{,} = 0$ and L unites the curve with radius $r_{,,}$ with a tangent and the second parenthesis = 0, whence

$$L = x_{,,}\left(1 + \frac{x_{,,}^4}{40P^2} - \frac{x_{,,}^8}{1152P^4} + \frac{x_{,,}^{12}}{13312P^6} - \&c., \frac{x^n}{NP^{\frac{n}{2}}}\right. \quad (60)$$

EXAMPLE 10. Given $x_{,,} = 209.65$; $x_{,} = 60$; $r_{,,} = 818.8$; $r_{,} = 2865$, to find the length of the parabolic arc connecting the circular curves with radii $r_{,,}$ and $r_{,}$, by 59:

$$L = x_{,,}\left(1 + \frac{x_{,,}^2}{40r_{,,}^2}\right) - x_{,}\left(1 + \frac{x_{,}^2}{40r_{,}^2}\right) = 209.65\left(1 + \frac{(209\,65)^2}{40\,(818.8)^2}\right) - 60\left(1 + \frac{60^2}{40\,(2865)^2}\right) \therefore L = 210 - 60 = 150.$$

If $r_{,} = \infty$, $L_{,} = o$; $L = 210 = L_{,,}$.

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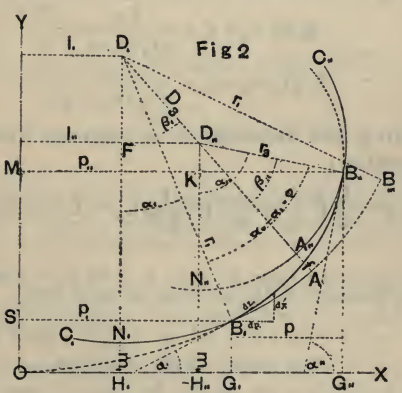
CHAPTER II.

If in equations 3, 5, 8, 15, 17, 28, 39, 41 obtained by the foregoing process we write L for ϕ (or x) since they have been assumed to be equal for small central angles, the resulting equations will give a much nearer approximation to the true transition curve for a central angle up to 15°, which is rarely exceeded, thus:



$$Lr = P; \qquad r_1 = \frac{P}{L_1} \qquad L_1 = \frac{P}{r_1} \qquad L_{11} = \frac{P}{r_{11}} \tag{61}$$

$$B_1B_{11} = L_{11} - L_1 = L = \left(\frac{P}{r_{11}} - \frac{P}{r_1}\right) = P\left(\frac{1}{r_{11}} - \frac{1}{r_1}\right) \tag{62}$$

which is the ideal equation of all transition curves.

EXAMPLE 11. Given $P = 171900$; $r_{II} = 818.8$; $r_I = 2865$, to find the length of the transition curve uniting the circular curves with radii r_{II} and r_I, by (62)

$$L = P\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right) = 171900\ (.00122 - .0035) = 171900 \times .00087 = 150.$$

If curves are reversed change the sign of $\frac{1}{r_I}$ from $-$ to $+$; then

$$L = L_{II} + L_I = 171900 \times .00157 = 210 + 60 = 270.$$

$$\frac{dy}{dL} = \frac{L^2}{2P}(= \sin\alpha);\ \therefore\ y = \int \frac{L^2 dL}{2P}; \qquad (63)$$

$$y = \frac{L^3}{6P};\ \text{and since}\ \frac{dx}{dL} = \cos\alpha,$$

$$x = \int dL\left(1 - \frac{L^4}{8P^2} + \right) = L\left(1 - \frac{L^2}{40\,r_{II}^2} + \right); \qquad (64)$$

$$B_I D_I N_I = \alpha_I = \frac{L r_{II}}{2 r_I (r_I - r_{II})};$$

$$B_{II} D_{II} N_{II} = \alpha_{II} = \frac{L r_I}{2 r_{II} (r_I - r_{II})}; \qquad (65)$$

subtracting and reducing to a common denominator and factoring:

$$\alpha_{II} - \alpha_I = \phi = \frac{L}{2}\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right) = \frac{1}{2P}(L_{II}^2 - L_I^2) = \beta_{II} + \beta_I; \qquad (66)$$

EXAMPLE 12. Given $L = 150$; $r_{II} = 818.8$; $r_I = 2865$, to find the angle subtended by the arc of the spiral, by (66):

$$\alpha_{II} - \alpha_I = \beta_{II} + \beta_I = \frac{L}{2}\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right) = \text{arc}\ \phi =$$

$$75 \times .00157 = .11175 \quad \phi^\circ = 6^\circ\ 45'.$$

For $r_I = \infty \frac{1}{r_I} = 0$, $\beta_I = 0$, $\text{arc}\ \phi = \frac{L}{2 r_{II}} = \text{arc}\ \beta_{II} = .0915$, $\beta_{II}^\circ = 5^\circ\ 15'$.

For $r_{II} = \infty \frac{1}{r_{II}} = 0$, $\beta_{II} = 0$, $\text{arc}\ \phi = \frac{L}{2 r_I} = \text{arc}\ \beta_I = .02625$, $\beta_I^\circ = 1^\circ\ 30'$.

$$N_{II} H_{II} = m_{II} = \frac{L^2 r_I^2}{24 r_{II} (r_I - r_{II})};$$

$$N_I H_I = m_I = \frac{L^2 r_{II}^2}{24 r_I (r_I - r_{II})}; \qquad (67)$$

$$y - y_{\prime} = \frac{L^2 (r_{\prime} - r_{\prime\prime})}{48\, r_{\prime} r_{\prime\prime}} = y_{\prime\prime} - y = \tfrac{1}{2} f; \qquad (68)$$

$\frac{1}{2}f$ = the ordinate of the middle point of L.

$$\mathbf{f} = \frac{\mathbf{L}^2}{\mathbf{24}} \left(\frac{\mathbf{1}}{\mathbf{r}_{\prime\prime}} \mp \frac{\mathbf{1}}{\mathbf{r}_{\prime}} \right) = \mathbf{A}_{\prime}\mathbf{A}_{\prime\prime} \qquad (69)$$

in which $L = P \left(\frac{1}{r_{\prime\prime}} \mp \frac{1}{r_{\prime}} \right) = B_{\prime} B_{\prime\prime}$.

EXAMPLE 13. Given $L = 150$; $r_{\prime\prime} = 818.8$; $r_{\prime} = 2865$, to find $f = A_{\prime} A_{\prime\prime}$, by (69):

$$f = \frac{L^2}{24} \left(\frac{1}{r_{\prime\prime}} - \frac{1}{r_{\prime}} \right) = \frac{(150)^2}{24} (.00122 - .00035) = \frac{(150)^2}{24} \times .00087;$$

$$f = \frac{22500}{24} \times .00087 = .815$$

For reverse curves the perpendicular distance between the tangents to the circular curves, produced, which are parallel to a tangent to the common origin of the transition curves is expressed by

$$f_{\prime\prime} + f_{\prime} = \frac{1}{24} \left(\frac{L_{\prime\prime}^2}{r_{\prime\prime}} + \frac{L_{\prime}^2}{r_{\prime}} \right). \quad \text{If } L_{\prime\prime} = L_{\prime} \text{ then } f_{\prime\prime} + f_{\prime} = \frac{L^2}{24} \left(\frac{1}{r_{\prime\prime}} + \frac{1}{r_{\prime}} \right)$$

In this and all examples given on the above data, it is to be observed that the value (d) which will appear in subsequent investigation is inappreciable and need not be regarded, also that the cubic parabola and spiral are practically the same and the formulas of either may be used indiscriminately. Table (2) may be regarded without serious error as adapted to either case.

If the curves $r_{\prime\prime} r_{\prime}$ are reversed, use the + sign; if in the same direction, use the − sign in equation (69).

Solving (69) for $\frac{1}{2}L$ we have

$$\frac{1}{2}L = \sqrt{6f \div \left(\frac{1}{r_{,,}} \mp \frac{1}{r_{,}}\right)}, \qquad (70)$$

which is Froude's curve of adjustment. The above equations may be obtained by an independent method, as will be shown further on, where the values of x, y and f, respectively will appear as the first term of a series. In all the above equations if $r_{,}$ be made equal to infinity, the terms in which it appears become = 0, the terms containing it disappear, and the resulting equations are for uniting a circular curve, with radius $r_{,,}$ with a tangent, by means of the transition curve. If the sign of $r_{,}$ be changed and $L_{,,}$ and $L_{,}$ be substituted for L in connection with $r_{,,}$ and $r_{,}$, the equations will be those for uniting curves reversed, by means of the transition curve, so that (61-69) inclusive, are general equations for the transition spirals and are sufficiently accurate when they are used for computing spirals for transition purposes only.

§ 11. Tables 1 and 2 are prepared by the use of equations 62, 63, 64, 66 and 69, in which $\frac{1}{r_{,}}$ is made zero, except where the value of x is considerable, when 115 is used.

A table should be computed consistent with the character of the alignment to which it is applied, of which table 2 may be taken as an example where $\frac{1}{r_{,}}$ in the formulas has been made = zero.

It is to be observed that the superelevation of the outer rail, in the use of the transition curve, may be

made greater or less than that which has been assumed in computing the tables; the only effect it will have is to diminish or increase the assumed value of "i", which is equivalent to increasing or diminishing the velocity, since i and v may be made variables in the constant P, i. e., it makes the rate of rise of the outer rail to effect superelevation a little greater or less. It is, however, best to introduce the average velocity of the express or fast passenger trains in constructing the tables. Where the location is so constrained that the PT's and PC's of the circular curves are quite close together, it may be necessary to give "i" a smaller value than would be otherwise desirable. A value of 300 or 400 is sufficient for adjustment, and good results may be obtained with a value of 200 when the radius is not greater than 573 feet. Equations (62-69) contain all the elements necessary to determine a suitable curve if intended only for the purpose of adjustment, and on account of their simplicity are recommended for ordinary use. The latter remark presupposes that Problem III will be used when the distance assunder (or f) of the two circular curves to be united is much in excess of that required for a transition curve with the fixed value of P.

§ **12.** If at the point B_1 (see Fig. 2) we imagine r_1 to increase till it becomes equal to infinity, the curvature of $B_1A_1 = 0$ and the arc B_1A_1 will be a straight line still preserving its tangency to the transition curve. The curvature at B_{11} will diminish to the same extent, i. e., the *difference* between curvature at B_1 and B_{11} will be the same as when $\beta_1 = 0$, and $\beta_{11} = \frac{L^2}{2P}$. The ordinates x and y can be computed and laid off from the new tangent as axis of abscissa with B_1 as origin the same as if from O. If we now conceive this new axis of x to be curved to a radius r_1 the curvature of the transition curve at any point will

be increased by the same amount and the ordinates may, without serious error, be laid off normal to the arc $B_{,}A_{,}$ and establish points of the transition curve. The same reasoning will apply if $r_{,,} = \infty$ and $B_{,,}A_{,,}$ becomes a tangent and values of x and y be laid off from it with $B_{,,}$ as origin, except that the resulting transition curve would be convex to $B_{,,}A_{,,}$. The ordinates would, however, be equal to those of the corresponding distance from B. If $r_{,,}$ now resume its original length the curvature of the transition curve at any point will equal that of the circular curve with radius $r_{,,}$ *minus* the curvature it had in a contrary direction when $r_{,,}$ was equal to infinity and $B_{,,}A_{,,}$ a straight line. In determining $\beta_{,,}$ and $r = \frac{P}{L}$, data may be taken from the tables.

§ 13. To conform to the case under consideration

$$x = L\,(1 - \&c.),\quad y = \frac{L^3}{6\,P}(1 - \&c.)$$

x will be measured on $A_{,}B_{,,}$ and y normal to $A_{,}B_{,,}$. Equations for x and y are equally true whether the origin be taken as O, $B_{,,}$ or $B_{,,}$. To lay the transition curve out by the expressions for x and y, their values may be laid out simultaneously with corresponding equal chord measurements along the transition curve. It will be seen that the above values of x and y differ from those of the cubic parabola slightly (which attains its minimum radius when the central angle becomes $= 24^\circ\,6'$) and fulfill the conditions of a spiral whose curvature and length vary inversely as the radius, and may become infinitely great and the corresponding radius infinitely small or equal o. As this extreme is never reached in practice, its discussion is beyond the scope of this paper and will not be considered here.

§ 14. The principle enunciated in the paragraph preceding the above, enable us to prepare a table of deflection angles according to the following method, which may be more convenient than that laid down in Chapter I.

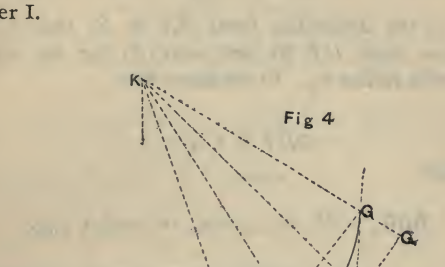


Fig 4

Refering to Fig. 4. If the deflection angles from AX to any point be denoted by $\delta\,\delta_{,}\,\delta_{,,}$ it will be found by computation that any angle as

$$\mathbf{DAX} = \delta_{,,} = \frac{\alpha_{,,}}{3} = \frac{\mathbf{L}}{6r_{,}} \text{ (nearly)}, \qquad (71)$$

in which $r_{,,} = DK$.

$$\mathbf{ADC} = \mathbf{DCX} - \mathbf{DAX} = \alpha_{,,} - \frac{\alpha_{,,}}{3} = \frac{\mathbf{L}}{2r_{,,}} - \frac{\mathbf{L}}{6r_{,,}} = \frac{\mathbf{L}}{3r_{,,}} = \qquad (72)$$

the angle which CT, a tangent common to the spiral and circular curve with radius $r_{,,}$ makes with the chord AD. To establish the points E, F and G by deflections at D

from tangent DT we have, from the paragraph already referred to,

$$EDT = \delta + \triangle, \tag{73}$$

and δ = the deflection from AX to B, and $\triangle$ = the deflection from DT to the point E, for the circular curve with radius $r_{,,}$. In the same way

$$GDT = \delta_{,,} + \triangle_{,,}; \tag{74}$$

the angle

$$\mathbf{DGG}_{,,} = 2\delta_{,,} + \triangle_{,,} = (\alpha_{,,} - \alpha_{,}) - (\delta_{,,} + \triangle_{,,}). \tag{75}$$

If we add $(\delta_{,,} + \triangle_{,,})$ to both members of the equation, we have:

$$\mathbf{GOT} = \alpha_{,,} - \alpha_{,} = \phi = 3\,\delta_{,,} + 2\,\triangle_{,,}. \tag{76}$$

in which $\triangle_{,,} = L\triangle$, in which $\triangle$ = the deflection for a unit length of DG on the circular curve with radius $r_{,,}$ and L the units from D of any point laid off on DG.

The above values are approximate. If close results are indispensable, the method indicated in connection with Fig. 3, in determining $\delta\,\delta_{,}\,\delta_{,,}$ will have to be resorted to.

§ 15. As indicated on page 20, we now proceed to verify certain equations (3-41, 61-69).

We have from the calculus:

$$r\,d\,\alpha = dL, \quad L = \frac{P}{r} = Pr^{-1}, \tag{77}$$

differentiating

$$dL = Pr^{-2}dr, \quad r\,d\,\alpha = Pr^{-2}dr, \quad d\,\alpha = \frac{Pr^{-2}\,dr}{r} = Pr^{-3}dr; \tag{78}$$

$$\alpha = \int Pr^{-3}\,dr = \frac{Pr^{-2}}{2} = \frac{P}{2r^2}, \tag{79}$$

hence for any two angles α_{II} and α_I we have:

$$\alpha_{II} = \frac{P}{2r_{II}^2}, \quad \alpha_I = \frac{P}{2r_I^2}; \tag{80}$$

$$\alpha_{II} - \alpha_I = \frac{P}{2r_{II}^2} - \frac{P}{2r_I^2} = \frac{P}{2}\left(\frac{1}{r_{II}^2} - \frac{1}{r_I^2}\right), \tag{81}$$

or

$$\alpha_{II} - \alpha_I = \frac{P}{2}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right). \tag{82}$$

From (62), $L_I = \frac{P}{r_I}$, $L_{II} = \frac{P}{r_{II}}$; whence

$$L_{II} - L_I = L = P\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right); \quad \frac{L}{2} = \frac{P}{2}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right); \tag{83}$$

substituting in value for $\alpha_{II} - \alpha_I$, we have

$$\alpha_{II} - \alpha_I = \frac{\mathrm{L}}{2}\left(\frac{1}{\mathrm{r}_{II}} \pm \frac{1}{\mathrm{r}_I}\right) = \phi = \beta_{II} \pm \beta_I; \tag{84}$$

the + sign being used when *curves are in the same direction* and *the − sign* when the *curves are in a reversed direction.* (If the − sign is used the second member of (84) will still equal $\alpha_{II} - \alpha_I$, in which α_{II} is on the opposite side of the axis of X from α_I, or the curves are reversed, $L = L_{II} + L_I$, $\alpha_{II} = \beta_{II}$ and $\alpha_I = \beta_I$) in which

$$\phi = \beta_{II} + \beta_I \tag{85}$$

is the angle subtended by the arc L of the *spiral* as well as the sum of the circular arcs $A_I B_I + A_{II} B_{II}$, or of a circular arc whose length is $= L$ and radius $r = \dfrac{2}{\dfrac{1}{r_{II}} + \dfrac{1}{r_I}}$.* (86)

---

*$\phi = \frac{L}{2}\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right)$; $L = r\phi$, in which r = some length of radii that with a central angle ϕ will give an arc $= L$ or $L = r\phi$, L and ϕ having the same value as in equation 84. Then $\phi = \frac{L}{r} = \frac{L}{2}\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right)$; $\frac{1}{r} = \frac{1}{2}\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right)$, solving for $r = \dfrac{2}{\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right)}$ $\frac{1}{r_{II}} + \frac{1}{r_I} = \frac{2}{r}$ ∴ the reciprocal of r is an arithmetical mean of the reciprocals of r_{II} and r_I.

If in equation (84) we make $r_{,,} = \infty$, and $\beta_{,,} = 0$, then

$$\frac{L}{2r_{,}} = \beta_{,}, \quad \frac{L}{2} = r_{,}\beta_{,};$$

similarly we may find

$$\frac{L}{2r_{,,}} = \beta_{,,}, \quad \frac{L}{2} = r_{,,}\beta_{,,};$$

whence

$$L = \beta_{,,}r_{,,} + r_{,}\beta_{,}. \tag{87}$$

The angles subtended by $\beta_{,,}$ and $\beta_{,}$ arcs of spirals in this equation, are not precisely the same as result from the investigation following. In the value L, $\beta_{,,}r_{,,}$ and $\beta_{,}r_{,}$ are assumed to be equal as would appear from Nordling, and $A_{,,}A_{,}$, the shortest distance, to be at the extremity of equal arcs $A_{,,}B_{,,}$ and $A_{,}B_{,}$. For large angles, this is not the case there is a difference $= d$; a radius $r_{,}$ would not pass through the center of $r_{,,}$ exactly. In L_C the angle is divided properly. $A_{,}B_{,} < A_{,,}B_{,,}$ (see § 16.)

If $\beta_{,,}$ and $\beta_{,}$ be the angles subtended by the circular arcs, at radius = 1, then

$$L_C = \beta_{,,}r_{,,} + \beta_{,}r_{,}; \tag{88}$$

$\beta_{,,}r_{,,}$ and $\beta_{,}r_{,}$ being respectively the arcs of circles $A_{,,}B_{,,}$ and $A_{,}B_{,}$.

§ 16. It has been assumed, in the case of the cubic parabola, that the length of the arc of the transition curve does not differ sensibly from the sum of the lengths of the arcs of the two circles equal to each other with radii $r_{,,}$ and $r_{,}$, described from the ends of the transition curve to points where their circumferences approach nearest each other and their tangents are parallel, also that the abscissas of their centers equaled ½ the abscissa of the point at which the circle and cubic parabola become tangent to each other and have a common radius. In case the central angle or length is considerable, as may occur in the use of the spiral, the error arising from this supposition may not be disregarded and the true value of $A_{,,}B_{,,}$ (see Fig. 2) and $A_{,}B_{,}$ should be computed in determining the length of arc to lay off from $A_{,,}$ or $A_{,}$ to establish the respective points $B_{,,}$ and $B_{,}$. The arc of the circle having the longer radius, as $r_{,,}$, will always be the shorter.

The rectangular co-ordinates of the center of the circle with radius $r_{,}$, as $D_{,}$, referred to the origin O are: $OH_{,} = l_{,}$ and $H_{,}D_{,} = k_{,}$, and of $D_{,,}$, $OH_{,,} = l_{,,}$, $H_{,,}D_{,,} = k_{,,}$; whence

$$\frac{l_{,,} - l_{,}}{k_{,,} - k} = \tan(\alpha_{,} + \beta_{,}); \tag{89}$$

$$l_{II} = p_{II} - r_{II} \sin \alpha_{II}, \qquad l_I = p_I - r_I \sin \alpha_I; \tag{90}$$

$$k_{II} = r_{II} + m_{II}, \qquad k_I = r_I + m_I; \tag{91}$$

whence

$$\tan (\alpha_I + \beta_I) = \frac{(p_{II} - r_{II} \sin \alpha_{II}) - (p_I - r_I \sin \alpha_I)}{(r_I + m_I) - (r_{II} + m_{II})}, \tag{92}$$

or if we have the value of f

$$\cos (\alpha_I + \beta_I) = \frac{(r_I + m_I) - (r_{II} + m_{II})}{r_I - (r_{II} + f)}, \tag{93}$$

from either of these equations

$$\beta_{II}^\circ = \alpha_{II}^\circ - (\alpha_I^\circ + \beta_I^\circ), \tag{94}$$

with β_{II}° reduced to arc, then

$$r_{II} \beta_{II} = A_{II} B_{II} \tag{95}$$

and

$$\beta_I = \alpha_{II} - (\alpha_I + \beta_{II}), \quad r_I \beta_I = A_I B_I; \tag{96}$$

denoting the difference between L and $A_{II} B_{II} + A_I B_I$ by "d" we have

$$d = L - (A_{II} B_{II} + A_I B_I) = L - (r_{II} \beta_{II} + r_I \beta_I). \tag{97}$$

Placing

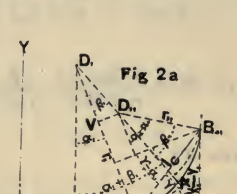
$$(r_{II} \beta_{II} + r_I \beta_I) = L_c, \tag{98}$$

$$\text{"d"} = L - L_c. \tag{99}$$

The point B_{II} should be marked with the name of the station at B_I + the computed length of the transition curve (as if $A_I B_I$ and $A_{II} B_{II}$ were laid off in the same manner as semi-tangent of a simple circular curve).

LIBRARY

To determine the angle β (second method): Let $\Upsilon = B_{II}\, B_I\, S_I$ = the angle that the chord $B_I\, B_{II}$ makes with the axis of X, then (Fig. 2*a*):



$\tan \Upsilon = \frac{y_{II} - y_I}{x_{II} - x_I}$. The angle which a tangent to the curve at B_I makes with the chord $B_I\, B_{II} = l_C$ is $\Upsilon - \alpha_I$. The chord $B_I\, B_{II} = l_C = \sqrt{(y_{II} - y_I)^2 + (x_{II} - x_I)^2}$. The distance $D_{II}\, V = l_C \cos (\Upsilon - \alpha_I) - r_{II} \sin (\alpha_{II} - \alpha_I)$. The distance $D_I V = r_I - [l_C \sin (\Upsilon - \alpha) + r_{II} \cos (\alpha_{II} - \alpha_I)]$.

$$\frac{D_{II}\, V}{D_I V} = \tan \beta_I = \frac{l_C \cos (\Upsilon - \alpha_I) - r_{II} \sin (\alpha_{II} - \alpha_I)}{r_I - [l_C \sin (\Upsilon - \alpha_I) + r_{II} \cos (\alpha_{II} - \alpha_I)]},$$

$$\beta_{II} = (\alpha_{II} - \alpha_I) - \beta_I.$$

$$H_I\, D_I\, A_I = \alpha_I + \beta_I = \alpha_{II} - \beta_{II},$$

transposing and changing signs,

$$\alpha_{II} - \alpha_I = \beta_{II} + \beta_I; \qquad (100)$$

§ 17. It remains to determine the distance

$$D_I\, D_{II} = r_I - (r_{II} + f) \qquad (101)$$

To do this we must first find an expression for f (and also for x and y) which is a line normal to both circular curves, joining the points at which they approach nearest each other; this is always on a line passing through the centers of the two circles; passing to rectangular co-ordinates, by the calculus, we have since, equation (13):

$$\frac{dy}{dx} = \tan\phi = \frac{\sin\phi}{\cos\phi}\frac{dL}{dL};^{*} \qquad (102)$$

$$dy = dL\sin\phi; \qquad (103)$$

$$dx = dL\cos\phi; \qquad (104)$$

in which ϕ = any angle. By trigonometry,

$$\sin\phi = \frac{\phi}{1} - \frac{\phi^3}{1.2.3} + \frac{\phi^5}{1.2.3.4.5} - \text{etc} \ldots\ldots \frac{\phi^n}{1.2.3..n};^{\dagger} \qquad (105)$$

$$\cos\phi = 1 - \frac{\phi^2}{1.2} + \frac{\phi^4}{1.2.3.4} - \text{etc} \ldots\ldots \frac{\phi^m}{1.2.3..m}; \qquad (106)$$

in which m may have any value from zero to infinity, n may have any value from one to infinity and ϕ any value from zero to infinity.

If in the equation $\phi = \frac{L}{2}\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right)$ we write for convenience $\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right) = \frac{1}{R}$, then $\phi = \frac{L}{2R} = \frac{L^2}{2P}$‡ which value in equations for $\sin\phi$ and $\cos\phi$ give, by (103):

$$\frac{dy}{dL} = \sin\phi = \frac{L^2}{2P}\left(1 - \frac{L^4}{24P^2} + \frac{L^8}{1920P^4} - \text{etc.}\right) = \frac{L}{2R}\left(1 - \frac{L^2}{24R^2} + \frac{L^4}{1920R^4} - \text{etc.}\right); \qquad (107)$$

---

*It should be remembered that L is an increasing function of x and y, and for any value imagined for dL, dx or dy undergoes a corresponding change.

†This and the succeeding formula are for expressing the trigonometrical function of an arc in terms of the arc itself (see Chauvenet's Trigonometry, Chapter XIII, 1867).

‡$a_{II} - a_I = \frac{P}{2}\left(\frac{1}{r_{II}^2} - \frac{1}{r_I^2}\right) = \phi$; make $r_I = \infty$, then $a_I = 0$, $a_{II} = \frac{P}{2r_{II}^2} = \phi$, and $\frac{1}{R} = \frac{1}{r_{II}}$, so $\phi = \frac{P}{2R^2} = \frac{LR}{2R^2} = \frac{L}{2R} = \frac{L^2}{2P}$.

EXAMPLE 14. Given $r_{II} = 818.8$; $r_I = 2865$; $L = 150$, to find ϕ.

$$\sin\phi = \sin(\beta_{II} + \beta_I) = \frac{L}{2}\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right)\left[1 - \frac{L^2}{24}\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right)^2 + \text{etc.}\right]$$

If we make $r_I = \infty$, $\frac{1}{r_I} = 0$, $\beta_I = 0$; we have approx

$$\sin\beta_{II} = \frac{150}{2 \times 818.8}\left(1 - \frac{(150)^2}{24 \times (818.8)^2} + \right) = .0916\,(1 - .001398) =$$
$$.0916 \times 99866 = .091477 \therefore \sin\beta_{II} = \sin 5°\,15'.$$

In the same way, if we make $r_{II} = \infty$, $\frac{1}{r_{II}} = 0$, $\beta_{II} = 0$, we shall find that $\sin\beta_I = .02618 = \sin 1°\,30' \therefore \beta_{II} + \beta_I = 5°\,15' + 1°\,30' = 6°\,45'$. $\beta_{II} - \beta_I = 3°\,45'$ = the change in direction of the tangents to the extremities of the transition curve when the main curves are reversed.

$$dy = dL\left(\frac{L^2}{2P} - \frac{L^6}{48P^3} + \frac{L^{10}}{3840P^5} - \text{etc.}\ldots\ldots \frac{L^n}{NP^{\frac{n}{2}}}\right); \quad (108)$$

integrating,

$$y = \frac{L^3}{6P}\left(1 - \frac{L^4}{56P^2} + \text{etc.}\ldots\ldots \frac{L^{n-2}}{NP^{\frac{n}{2}-2}}\right); \quad (109)$$

or in terms of R, since $P = RL$, $\frac{1}{R} = \frac{1}{r_{II}} - \frac{1}{r_I}$.

$$y = \frac{L^2}{6R}\left(1 - \frac{L^2}{56R^2} + \text{etc.}\right) =$$
$$\frac{L^2}{6}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)\left[1 - \frac{L^2}{56}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)^2 + \text{etc.}\right]; \quad (110)$$

EXAMPLE 15. Given $r_I = 2865$; $r_{II} = 818.8$; $L = 150$, to find y by (110):

$$y = \frac{L^2}{6}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)\left[1 - \frac{L^2}{56}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)^2 + \text{etc.}\right] =$$
$$\frac{(150)^2}{6}(.00122 - 00035)\,[1 - \frac{(150)^2}{56}(.00122 - .00035)^2 + \text{etc.}]$$
$$y = \frac{22500 \times .00087}{6}[1 - 22500\,(.00087)^2 + \text{etc.}] = 3.2625\,[.9997] = 3.26225.$$

or in terms of ϕ, since $L = 2R\phi$,

$$y = \frac{2}{3}R\phi^2\left(1 - \frac{\phi^2}{14} + \right) = \frac{\frac{2\phi^2}{3}}{\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)}\left[1 - \frac{\phi^2}{14} + \text{etc.}\right]; \quad (111)$$

similarly,

$$\frac{dx}{dL} = \cos\phi = \left(1 - \frac{L^4}{8P^2} - \frac{L^8}{384P^4} - \text{etc.}\right) = \left(1 - \frac{L^2}{8R^2} + \frac{L^4}{384R^4} - \text{etc.}\right); \quad (112)$$

$$dx = dL\left(1 - \frac{L^4}{8P^2} + \frac{L^8}{384P^4} - \frac{L^{12}}{46080P^6} + \text{etc.} \ldots\ldots \frac{L^m}{MP^{\frac{m}{2}}}\right) \quad (113)$$

integrating,

$$x = L\left(1 - \frac{L^4}{40P^2} + \frac{L^8}{3456P^4} - \frac{L^{12}}{599040P^6} + \text{etc.} \ldots\ldots \frac{L^m}{MP^{\frac{m}{2}}}\right); \quad (114)$$

$$x = L\left(1 - \frac{L^2}{40R^2} + \frac{L^4}{3456R^4} - \frac{L^6}{599040R^6} + \text{etc.}\right); \quad (115)$$

or in terms of ϕ, since $L = 2R\phi$,

$$x = 2R\phi\left(1 - \frac{\phi^2}{10} - \frac{\phi^4}{216}\right) = \frac{2\phi}{\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)}\left(1 - \frac{\phi^2}{10} - \frac{\phi^4}{216} - \text{etc.}\right) \quad (116)$$

also

$$x = L\left(1 - \frac{\phi^2}{10} + \frac{\phi^4}{216} - \text{etc.}\right) \quad (117)$$

$$L = \frac{x}{1 - \frac{\phi^2}{10} + \frac{\phi^4}{216} - \text{etc.}}. \quad (118)$$

$$\mathbf{x = L\left[1 - \frac{L^2}{40}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)^2 + \frac{L^4}{3456}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)^4 - \text{etc.}\right]} \quad (119)$$

EXAMPLE 16.

$$x = L\left[1 - \frac{L^2}{40}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)^2 \pm \text{etc. (other terms)}\right] = 150\left(1 - \frac{22500 \times .0000007569 + \pm}{40}\right);$$

$$x = 150\ (.9996 \pm) = 149.94.$$

It is to be observed that the above values of x and y do not differ appreciably from the values given by (8) and (64); the difference,

however, becomes greater as L and $\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right)$ become greater. x is laid off on the arc of the circle $A_{,}B_{,}$ or $A_{,,}B_{,,}$ (with radii $r_{,}$ and $r_{,,}$) as axis of X with $B_{,}$ or $B_{,,}$ as origin, x being the abscissa of y which is laid off normal, or radial, to the arc $A_{,}B_{,}$ or $A_{,,}B_{,,}$.

In general, by Fig. (2), any distance

$$m = y - R \operatorname{ver} \phi = y - R(1 - \cos \phi), \qquad (120)$$

in which R depends on $\frac{1}{r_{,,}} - \frac{1}{r_{,}}$ for its value, substituting for $\cos \phi$ its value, remembering that $R = \frac{P}{L}$, writing f for m and reducing, since

$$R(1 - \cos \phi) = R\left[1 - \left(1 - \frac{L^4}{8P^2} + \frac{L^8}{384P^4} - \text{etc.}\right)\right]; \quad (121)$$

$$f = y - R \operatorname{ver} \phi = L\left[\left(\frac{L^2}{6P} - \frac{L^6}{336P^3} + \text{etc.}\right) - \left(\frac{L^2}{8P} - \frac{L^6}{384P^3} + \text{etc.}\right)\right]; \qquad (122)$$

$$f = L\left(\frac{L^2}{24P} - \frac{L^6}{2688P^3} + \text{etc.}\right) = \frac{L}{P}\left(\frac{L^2}{24} - \frac{L^6}{2688P^2} + \text{etc.}\right); \quad (123)$$

since $\frac{L}{P} = \frac{1}{R}$,

$$f = \frac{L^2}{24R}\left(1 - \frac{L^2}{112R^2} + \text{etc.}\right), \text{ and since } \frac{1}{R} = \left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right), \quad (124)$$

$$f = \frac{L^2}{24}\left(\frac{1}{r_{,,}} \pm \frac{1}{r_{,}}\right)\left[1 - \frac{L^2}{112}\left(\frac{1}{r_{,,}} \pm \frac{1}{r_{,}}\right)^2 + \text{etc.}\right]. \quad (125)$$

EXAMPLE 17. Given $r_{,,} = 818.8$; $r_{,} = 2865$; $L = 150$; to find the offset f. 1st, when the circular curves $r_{,,}$ and $r_{,}$ turn in the same direction, by (125):

$$f = \frac{L^2}{24}\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right)\left[1 - \frac{L^2}{112}\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right)^2\right] =$$

$$\frac{(150)^2}{24} \times .00087\left[1 - \frac{(150)^2\,(.00087)^2}{112}\right] = .815$$

2nd, when the circular curves turn in opposite direction or are reversed. Given $r_{II} = 818.8$; $r_I = 2865$; $L_{II} = 210$; $L_I = 60$; to find the offset between the tangents to the circular curves which are parallel to the tangent to the transition curve which passes through its origin (Fig. 9).

$$f = f_{II} + f_I = \frac{1}{24}\left[\left(\frac{L_{II}^2}{r_{II}} + \frac{L_I^2}{r_I}\right) - \frac{1}{112}\left(\frac{L_{II}^4}{r_{II}^3} + \frac{L_I^4}{r_I^3}\right)\right] =$$

$$\frac{1}{24}[(53.86 + 1.25) - (.0000007)];$$

$$f = f_{II} + f_I = \frac{55.11}{24} = 2.29. \quad \text{If } L_{II} = L_I, \text{ then}$$

$$f = f_{II} + f_I = \frac{L^2}{24}\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right)\left[1 - \frac{L^2}{112}\left(\frac{1}{r_{II}} + \frac{1}{r_I}\right)^2\right].$$

If now we substitute for f its value in the equation for $D_I D_{II}$ we have for the distance between the centers of the two curves,

$$* \; \mathbf{D_I D_{II}} = \mathbf{r_I} - \left[\mathbf{r_{II}} + \frac{\mathbf{L}^2}{\mathbf{24}}\left(\frac{\mathbf{1}}{\mathbf{r_{II}}} - \frac{\mathbf{1}}{\mathbf{r_I}}\right)\mathbf{1} - \text{etc.}\right]. \qquad (126)$$

EXAMPLE 18. Given $L = 150$; $r_I = 2865$; $r_{II} = 818.8$; to find the distance between the centers of curves with radii r_{II} and r_I. 1st, if curves turn in the same direction, (Fig. 2):

$$D_{II} D_I = r_I - \left[r_{II} + \frac{L^2}{24}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)(1 - \text{etc})\right] = r_I - (r_{II} + f) =$$

$$2865 - \left[818.8 + \frac{(150)^2}{24} \times .00087\right];$$

$$D_{II} D_I = 2865 - [818.8 + .815] = 2865 - 819.615 = 2045.385.$$

2nd, if curves are reversed (or turn in a contrary direction) (Fig. 9), then: If $L_{II} = 210$; $L_I = 60$; $r_I = 2865$; $r_{II} = 818.8$:

$$D_I D_{III} = \left[r_I + \left(r_{II} + \frac{1}{24}\left(\left(\frac{L_{II}^2}{r_{II}} + \frac{L_I^2}{r_I}\right) - \frac{1}{112}\left(\frac{L_{II}^4}{r_{II}^3} + \frac{L^4}{r_I^3}\right)\right)\right)\right]$$

$$\times \sec K_{II}\, D_I\, D_{III} = [r_I + (r_{II} + f_{II} + f_I)] \sec K_{II} D_{II}\, D_{III};$$

$$D_I D_{III} = [2865 + (818.8 + 2.29)] \sec K_{II}\, D_I\, D_{II} = 3686.09 \sec K_{II}\, D_I\, D_{III};$$

$$\tan K_{II}\, D_I\, D_{III} = \frac{x_{II} + x_I}{r_I + r_{II} + f_{II} + f_I}.$$

---

* Also $D_I D_{II} = D_I V \sec \beta_I$ (by Fig. 2a) and $f = r_I - (r_{II} + D_I V \sec \beta_I)$

§ 18. If, in the foregoing values of y, x, $\frac{dy}{dx}$, and f we omit all terms in the bracket after the first, we have

$$y = \frac{L^2}{6}\left(\frac{1}{r_{II}} \pm \frac{1}{r_I}\right); \qquad (127)$$

$$x = L = x\left(1 + \frac{x^4}{40P^2}\right); \qquad (128)$$

$$\frac{dy}{dx} = \frac{L^2}{2P} = \frac{L}{2}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right) = \tan\phi; \qquad (129)$$

$$f = \frac{L^2}{24}\left(\frac{1}{r_{II}} \pm \frac{1}{r_I}\right), \qquad (130)$$

the same as (~~5, 8 and 11~~ 63, 64, 69) or from which latter equation we find:

$$\frac{L}{2} = \sqrt{6f \div \left(\frac{1}{r_{II}} \pm \frac{1}{r_I}\right)}; \qquad (131)$$

identical with the Froud's curve of adjustment, as indicated by Prof. Rankine (Civil Engineering, Edition 1863). If in equations (127, 128, 129, 130), we write ϕ or x for L, we again have the equations for a cubic parabola the same as in Chapter I.

If the transition curve embraces a large central angle equations (110, 119 and 125) should be used.

$\frac{1}{2}f = \frac{L^2}{48}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right)$, and if in the value for y we write $\frac{1}{2}L$ for L and cube it, then

$$y = \frac{L^3}{6P} = \frac{L^3}{8} = \frac{L^3}{48P} = \frac{L^2}{48R} = \frac{L^2}{48}\left(\frac{1}{r_{II}} - \frac{1}{r_I}\right). \qquad (132)$$

Hence the ordinate $y = \frac{1}{2}f$ is at the middle point of the transition curve.

Many of the formulæ of [§ 15-17], while more nearly correct are too extended for field computation. Tables 3, 4, etc. are

prepared by the use of them, and similar ones should be resorted to for operation in the field when those deduced by [§ 10-11] are not close enough.

§ 19. If $r_I = \infty, \beta_I = 0, \alpha_I = 0$. $A_I B_I$ is a straight line $= x_{II} - r_{II} \sin \beta_{II} = x_{II} - r_{II} \sin \phi = x_{II} - r_{II} \sin \alpha_{II} = OH_{II}$. (Fig. 2.)

§ 20. The transition curve should be no longer than required for purposes of adjustment, considering the highest speed that can be safely attained upon the circular portions of the alignment to which it is applied. The longer it is the more it increases the perplexities of the trackman, who is unable to maintain the transition curve in its proper place wholly "by eye" or by their usual methods. The aid of an engineer is indispensible if the transition curve is of considerable length as may occur when it has been used for the purposes of securing a better fitting to the ground in location, irrespective of its use as an adjustment curve. This latter contingency can usually be avoided by introducing a circular curve of intermediate radius between those to be joined, and uniting this intermediate curve with the principal ones by means of the proper transition curves based on the assumed values entering the constant P. For the above reasons, it is believed that the use of the transition curve for other purposes than adjustment should not be encouraged. With these conditions adhered to, a table can be prepared giving the ordinates at regular intervals on the long chord or along the length of the transition curve.

The beginning and end of the transition curve should be marked by permanent points.

A few useful problems follow in the application of the transition curve.

PROBLEM I.

TO FIND THE SEMI-TANGENTS.

§ 21. Given a circular curve whose radius $= r_{,,}$; the intersection angle $= I$; the semi-tangent $= T$, to unite it with tangents by means of transition curves whose lengths are L, and L and offsets are f, and f respectively.

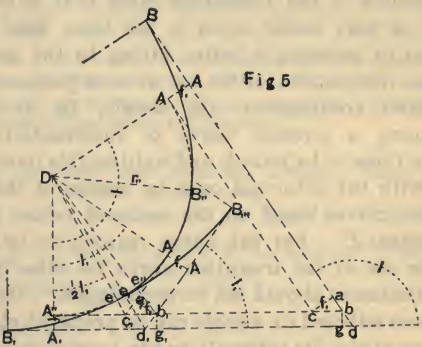


Fig 5

CASE I.

When f, > f (by Fig. 5), if $I < 90°$,

$$B_{,,,}d_{,} = B_{,,,}A + Ac_{,} - a_{,}c_{,} + b_{,}d_{,} = x_{f_{,}} + T - f_{,}\cot I + f\,\mathrm{cose}\,I. \qquad (1)$$

If $I > 90°$,

$$Bd = BA + Ac + ab + bd = x_{f_1} + T - f_1(-\cot I) + f \operatorname{cose} I; \quad (2)$$

$$Bd = x_{f_1} + T + f_1 \cot I + f \operatorname{cose} I; \quad (3)$$

or in general calling,

$$\mathbf{B_{III}\,d_1 \text{ or } Bd = t_1;\ t_1 = x_{f_1} + T \mp f_1 \cot I + f \operatorname{cose} I,} \quad (4)$$

the + sign being used when $I > 90°$ and the − sign

When $I < 90°$.

$$B_1 d_1 = B_1 A_1 + A_{11} c_1 + c_1 b_1 - d_1 g_1 = x_f + T + f_1 \operatorname{cose} I - f \cot I; \quad (5)$$

$$B_1 d = B_1 A_1 + A_{11} c + cb + dg = x_f + T + f_1 \operatorname{cose} I + f \cot I \quad (6)$$

In general calling,

$$\mathbf{B_1 d_1 \text{ or } B_1 d = t = x_f + T + f_1 \operatorname{cose} I \pm f \cot I,} \quad (7)$$

using + when $I > 90°$ and − when $I < 90°$.

CASE 2.

If $\mathbf{f = 0}$ in equation 4, we have

$$\mathbf{t_1 = T + f_1 \operatorname{cose} I, \quad t = x_f + T + f_1 \cot I.} \quad (8)$$

CASE 3.

If $\mathbf{f_1 = f}$ in equation 4,

$$t_1 = x_{f_1} + T + f_1(\operatorname{cose} I \pm \cot I),$$
$$t = x_f + T + f(\operatorname{cose} I \pm \cot I). \quad (9)$$

If $I < 90°$, the last term of t_1 is f_1 (cose I – cot I) and by trigonometry (see Chauvenet's, p. 35),

$$\text{cose } I - \cot I = \frac{1}{\sin I} - \frac{\cos I}{\sin I} = \frac{1 - \cos I}{\sin I} = \tan \tfrac{1}{2} I \therefore$$

$$t_1 = x_{f_1} + T + f_1 \tan \tfrac{1}{2} I \text{ or } t_1 = x_{f_1} + (r_{11} + f_1) \tan \tfrac{1}{2} I; \quad (10)$$

similarly, if $I > 90°$,

$$\text{cose } I + \cot I = \frac{1 + \cos I}{\sin I};$$

but if $I > 90°$,

$$\cos I = -\cos I$$

$$\text{cose } I + \cot I = \frac{1 - \cos I}{\sin I},$$

which we have seen = tan ½I. Hence **when $f_1 = f$,**

$$t_1 = t = x_{f_1} + T + f_1 \tan \tfrac{1}{2} I = x_{f_1} + (r_{11} + f_1) \tan \tfrac{1}{2} I \quad (11)$$

is a general equation whether I be greater or less than 90°.

EXAMPLE 19. Given $L = 210$; $r = 818.8$; $f = 2.23$; $I = 40°$; $OG_1 = 104.93$; to find $t =$ Fig. (5):

$t = B_1 A_1 + (r_{11} + f) \tan \tfrac{1}{2} I = (104.93 + 821.03) .36397 = 403.75.$

With the same data

TO FIND THE EXTERNAL SECANT.

CASE I.

When $f_1 > f$,

$$\frac{A_1 d_1}{D A_1} \text{ (or } A_1 d) = \frac{T + f_1 \text{ cose } I + f \cot I}{r_{11} + f} = \tan A_1 D d_1;$$

$$e_1 d_1 = (r_{11} + f) \sec A_1 D d_1 - r_{11}. \quad (12)$$

CASE 2.

If $f = 0$,

$$\frac{A_{,,} b_{,}}{r_{,,}} = \frac{T + f_{,} \operatorname{cose} I}{r_{,,}} = \tan A_{,,} D b_{,};$$

$$e_{,,} b_{,} = r_{,,} \sec A_{,,} D b_{,} - r_{,,}. \qquad (13)$$

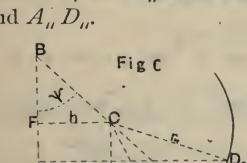
CASE 3.

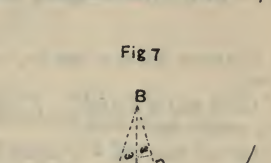
$$f_{,} = f, \; e_{,} d_{,} = (r_{,,} + f) \sec \tfrac{1}{2} I - r_{,,}. \qquad (14)$$

PROBLEM II.

TO FIND THE LOCATION OF THE OFFSET "*f*".

§ 22. Given two curves with radii $r_{,}$ and $r_{,,}$, a distance $D_{,} D_{,,} = d$ joining the points $D_{,}$ and $D_{,,}$, also the angles $B D_{,} D_{,,} = \rho$ and $C D_{,,} D_{,} = \theta$, to find the points $A_{,}$ and $A_{,,}$ at which a line drawn through the centers $r_{,}$ and $r_{,,}$ at B and C will cut the curves $A_{,} D_{,}$ and $A_{,,} D_{,,}$.





From Fig. 6 we have

$$FC = h = d \sin \rho - r_{,,} \sin (180 - (\rho + \theta)), \qquad (1)$$

and since $BC = r_{,} - (r_{,,} + f_{,})$,

$$\frac{FC}{BC} = \frac{h}{r_{,} - (r_{,,} + f)} = \frac{d \sin \rho - r_{,,} \sin (180 - (\rho + \theta)}{r_{,} - (r_{,,} + f)} = \sin \Psi \quad (2)$$

$$D_{,,} C D_4 = \Sigma = 180 - (\rho + \theta), \ \Psi - \Sigma = \omega = D_{,,} C A_{,,}; \quad (3)$$

whence

$$r_{,,} \omega = A_{,,} D_{,,}, \quad (4)$$

in which ω is an arc.

If $\Sigma > \Psi$, then the point $A_{,,}$ lies between $D_{,,}$ and D_3 and the distance $A_{,,} D_3$ is measured from D_3 towards $D_{,,}$ to $A_{,,}$.

If $\Psi > \Sigma$, the point $A_{,,}$ lies beyond $D_{,,}$ and the distance $A_{,,} D_{,,}$ is measured from $D_{,,}$ to $A_{,,}$ whence $A_{,,}$ is established. On a perpendicular to a tangent at $A_{,,}$ and from $A_{,,}$ lay off $f_{,}$ and establish $A_{,}$.

When f is small, the direction of the radial line can be estimated near enough. The method of fixing $B_{,}$ and $B_{,,}$, in Fig. 2, has already been indicated. If $r_{,} = \infty$, $D_{,} A_{,}$ is tangent. Fig. 7 applies.

EXAMPLE 20. $r_{,} = 1000$; $r_{,,} = 600$; $d = 300$; $\rho = 70°$; $\theta_{,} = 96°$; $f = 50'$:

1st to find $h = d \sin \rho - r_{,,} \sin (180 - (\rho - \theta_{1}) = 300 \times .93969 - 600 \times .24192 = 136.75 = FC$.

2nd to find $\sin \Psi = \frac{h}{r_{,} - (r_{,,} + f_{,})} = \frac{136.75}{1000 - (600 + 50)} = \frac{136\ 75}{350} = .3097$; $\frac{FC}{BC} = \sin 23°.0$.

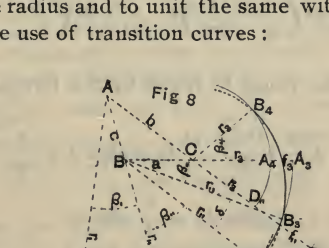
3rd to find $180 - [70° + 96°] = 14° = \Sigma$; $\Psi - \Sigma = \omega = 23° - 14° = 9°$; ω reduced to arc $= \frac{9}{57.29} = .1571$.

$r \omega = 600 \times .1571 = 94.26 = A_{,,} D_{,,}$, and since $\Psi > \Sigma$, $A_{,,} D_{,,}$ is measured from $D_{,,}$ towards D_3.

If $\Sigma = D_4 C D_3$, then $\Sigma > \Psi$ and $A_{,,} D_3$ would be measured from D_3 towards $D_{,,}$ to establish point $A_{,,}$; $A_{,}$ is on a perpendicular to a tangent to the curve with radius $r_{,,}$ at $A_{,,}$.

PROBLEM III.

§ 21. To unit two circular curves turning in the same direction, whose centers are a fixed distance apart, by the introduction of a third circular curve of an intermediate radius and to unit the same with the main curves by the use of transition curves:



Given two circular curves with radii r_1 and r_3, respectively, whose centers are apart a distance $AC = b = r_1 - (r_3 + f_{11})$, and which are separated from each other a distance $D_1 D_{11} = f_{11}$. It is desired to introduce between them a third curve with radius r_{11}, less than r_1 and greater than r_3 and to join the curve with radius r_{11} with those having radii r_1 and r_3 by means of transition curves:

By the figure $AD_1 = AA_1 = AB_1 = r_1 = AC + CD + D_{11} D_1 = b + r_3 + f_{11}$, whence

$$AC = b = r_1 - (r_3 + f_{11}); \qquad (1)$$

$$AB = c = r_{,} - (r_{,,} + f_{,}); \tag{2}$$

$$BC = a = r_{,,} - (r_{3} + f_{3}); \tag{3}$$

a, b and c form the sides of a triangle ABC in which

$$f_{,} = \frac{L_{,}^{2}}{24}\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right); \quad f_{3} = \frac{L_{3}^{2}}{24}\left(\frac{1}{r_{3}} - \frac{1}{r_{,,}}\right); \tag{4}$$

$$L_{,} = P\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right); \quad L_{3} = P\left(\frac{1}{r_{3}} - \frac{1}{r_{,,}}\right). \tag{5}$$

Any angle as A may be found by the formulæ,

$$\text{Ver } A = 2\frac{(s-b)(s-c)}{bc} \text{ in which } s = \frac{1}{2}(a+b+c; \tag{6}$$

$$\text{Sin } B = \frac{b}{a}\sin A; \tag{7}$$

$$\text{Sin } C = \frac{c}{a}\sin A; \tag{8}$$

Reducing each of the above angles to arc as indicated in another part of this book we have,

$$A_{,} D_{,} = r_{,} A; \; A_{4} D_{,,} = r_{3} C; \; A_{,,} A_{3} = r_{,,} (180 - B) \tag{9}$$

the arc

$$\beta_{4} = \frac{L_{3}}{2r_{3}} = \frac{P}{2r_{3}}\left(\frac{1}{r_{3}} - \frac{1}{r_{,,}}\right); \tag{10}$$

$$\beta_{3} = \frac{L_{3}}{2r_{,,}} = \frac{P}{2r_{,,}}\left(\frac{1}{r_{3}} - \frac{1}{r_{,,}}\right); \tag{11}$$

in the same way,

$$\beta_{,,} = \frac{L_{,}}{2r_{,,}} = \frac{P}{2r_{,,}}\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right); \tag{12}$$

$$*\beta_1 = \frac{L_1}{2r_1} = \frac{P}{2r_1}\left(\frac{1}{r_{11}} - \frac{1}{r_1}\right); \qquad (13)$$

$$A_1 B_1 = r_1 \beta_1; \quad A_{11} B_{11} = r_{11} \beta_{11}; \quad A_3 B_3 = r_{11} \beta_3;$$
$$A_4 B_4 = r_3 \beta_4; \qquad (14)$$

$$B_{11} B_3 = r_{11} \delta; \quad \delta = (180 - B) - (\beta_{11} + \beta_3); \qquad (15)$$

$$B_1 D_1 = r_1 A + r_1 \beta_1 = r_1 (A + \beta_1); \qquad (16)$$

$$D_{11} B_4 = r_3 (C + \beta_4). \qquad (17)$$

EXAMPLE 21. Given $r_1 = 2865$; $r_{11} = 1146$; $r_3 = 716.3$; $f_{11} = 20$; $P = 171900$; $L_1 = 90$; $L_3 = 90$ $\therefore f_1 = f_3 = .17$ to join the circular curves $B_1 D_1$ and $B_4 D_{11}$ by Prob. III. First finding a, b and c by (1–3) we have by (6) ver $A = 3°44'$; arc $A = .06516$; by (7) $(180 - B) = 18°48'$; arc $(180 - B) = .32811$; by (8) $C = 15°06'$; arc $C = .26054$; $A_1 D_1 = 2865 \times .06516 = 186.68$; $A_4 D_{11} = 716.3 \times .26054 = 186.77$; $A_{11} A_3 = 1146 \times .3281 = 376.00$; $\beta_4 = \frac{90}{2(716.3)} = .06282$; $\beta_3 = \beta_2 = \frac{90}{2(1146)} = .03926$; $\beta_1 = \frac{90}{2(2865)} = .0157$; $A_1 B_1 = A_{11} B_{11} = A_3 B_3 = A_4 B_4 = 45.0$; $\delta = 18°48' - (2°15' + 2°15') = 14°20'$; arc $\delta = .25017$; $B_{11} B_3 = 1146 \times .25017 = 286.7$; $B_1 D_1 = 2865\ (.06516 + .0157) = 231.67$; $B_4 D_{11} = 716.3\ (.26054 + .06282) = 231.62$.

If we make $f_1 = 0$ and $r_1 = r_{11}$, then $c = 0$, $C = 0$, f_{11} and $\beta_{11} = 0$, $b = a$ and is coincident with it, $A = 180 - B$ and the problem reduces to uniting $A_{11} A_3$ with $D_{11} B_4$ by means of the transition curve. If $r_{11} = r_1$, $c = 0$, $C = 0$, in which case f_1 and $f_3 = 0$; a is equal to and coincides with b and $A = 180 - B$, equations for vers $= 0$. The problem reduces to uniting curves $B_1 D_1$ and $B_4 D_{11}$ by means of a transition curve whose length $L_2 = \sqrt{\frac{24 f_{11}}{\frac{1}{r_1} - \frac{1}{r_3}}}$.

If f_3 and $f_1 = 0$ while r_1, r_{11} and r_3 retain their values, the transition curves disappear and the curve with radius r_{11} compounds at A_4 and A_1 with the curves having radii r_1 and r_3.

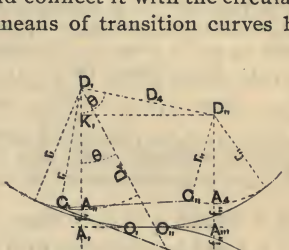
$$180 - B \text{ is the central angle of } A_{11} A_3; \qquad (18)$$

$$C = \text{that of } D_{11} A_4; \qquad A = \text{that of } A_1 D_1 \qquad (19)$$

*For determining large central angles of β_1, β_{11}, β_3 and β_4 the method of §16 is to be preferred.

PROBLEM IV.

§ 22. Given two curves turning in the same direction, whose centers are a distance apart $= D_1 D_{11}$ and whose radii are r_1 and r_{11}. To fix the position of a tangent $A_1 A_{111}$ and connect it with the circular curves $A_{11} C_1$ and $A_4 C_{11}$ by means of transition curves having a fixed



value of P (Fig. 9). Let $C_1 C_{11}$ be a line joining any two points C_1 and C_{11} of the circular curves with radii r_1 and r_{11}; $D_1 D_{11}$ the known distance between the centers of the curves r_1 and r_{11}. Measure the angles $D_1 C_1 C_{11}$ and $C_1 C_{11} D_{11}$. By traverse we find the angles $C_1 D_1 D_{11}$ and $C_{11} D_{11} D_1$. If from D_1 and D_{11} we let fall perpendiculars to any imaginary tangent passing through the origins of the transition curves, by the conditions of the problem the length of these perpendiculars will be:

$$D_1 A_1 = r_1 + f_1 \text{ and } D_{11} A_{111} = r_{11} + f_{11},$$
$$\text{and any distance } D_1 K_1 = (r_1 + f_1) - (r_{11} + f_{11}), \quad (1)$$

and if we denote the distance $D_{,} D_{,,}$ by D_4, then

$$\frac{D_{,} K_{,}}{D_{,} D_{,,}} = \cos \theta = \frac{(r_{,} + f_{,}) - (r_{,,} + f_{,,})}{D_4}. \qquad (2)$$

$C_{,} D_{,} A_{,,}$ = difference of $C_{,} D_{,} D_{,,}$ and θ. $C_{,} D_{,} A_{,,}$ reduced to arc and multiplied by $r_{,}$ = the distance $C_{,} A_{,,}$ to establish $A_{,,}$, from $A_{,,}$ lay off $f_{,}$ normal to the circle and establish $A_{,}$. In the same way $C_{,,} D_{,,} A_4$ = the difference between 180° − θ and $C_{,,} D_{,,} D_{,}$. $C_{,,} D_{,,} A_4$ multiplied by $r_{,,}$ = arc distance $C_{,,} A_{,,}$ from $C_{,,}$ to A_4 to establish A_4, from which lay off $f_{,,}$ and establish $A_{,,,}$; then a line through $A_{,} A_{,,,}$ is the required tangent.

The distance $A_{,} O_{,}$ to the origin $O_{,} = x_{,} - r_{,} \sin \alpha_{,}$ and the distance $A_{,,,} O_{,,} = x_{,,} - r_{,,} \sin \alpha_{,,}$; whence the distance

$$O_{,} O_{,,} = D_{,} D_{,,} \sin \theta - [(x_{,,} - r_{,,} \sin \alpha_{,,}) + (x_{,} - r_{,} \sin \alpha_{,})]. \quad (3)$$

If we wish the distance OO = zero, make

$$D_{,} D_{,,} = [(x_{,,} - r_{,,} \sin \alpha_{,,}) + (x_{,} - r_{,} \sin \alpha_{,})] \operatorname{cose} \theta. \quad (4)$$

This gives the shortest possible distance between the centers of the circular curves where the transition curves are introduced. The transition curves may have different values of P provided $A_{,} O_{,} + A_{,,,} O_{,,}$ = or < than $A_{,} A_{,,,}$. If the curves are reversed, $D_{,} K_{,,} = (r_{,} + f_{,}) + (r_{,,} + f_{,,})$ and we find the value of θ by completing the traverse $C_{,} C_{,,,} D_{,,,} D_{,} C_{,}$.

EXPLANATION OF THE TABLES.

BY RECTANGULAR CO-ORDINATES.

§ **25.** Tables I to IX give values for laying out the transition curve by the method of rectangular co-ordinates, They are equally applicable for uniting a tangent with a circular curve, or curves having different radii, by means of the transition curve. $L_{,,}$ and $L_{,}$ may be taken separately from the same column as also may $\alpha_{,,}$ and $\alpha_{,}$, and their difference will be the value of L and ϕ for the length and central angle respectively. The several ordinates, x, y, x_f, y_f, are laid off from $B_{,}$ or $B_{,,}$ as origin (Fig. 2) with arc $B_{,}A_{,}$ or $B_{,,}A_{,,}$ as axis,—the same as if $B_{,}$ or $B_{,,}$ were written for A in Fig. 3, and the successive stations were B, C, D, etc. and $B_{,}$, $C_{,}$, $D_{,}$, etc., successive points x, $x_{,}$, $x_{,,,}$, etc. with corresponding $y, y_{,}, y_{,,}$, etc., values normal to the curved axis $A_{,}B_{,}A_{,,}B_{,,}$ in the same manner as if $A_{,}$ $B_{,}$ were tangent. Values of d are dropped, as in no case in these tables do they reach $\frac{1}{10}$ of a foot. Table II is extended as an example to develop these differences to an amount worthy of consideration. P and v are taken of such values as to avoid introducing fractions in L. L is supposed to be measured on the curve, but since the chords are generally quite short, the sum of their lengths is but little less than that of the curve, hence no allowance is made for the length of the curve being in excess of the sum of the lengths of the chords.

EXAMPLE 1. Given $L = 120$; $r_I = 1432.5$; $r_{II} = 716.3$ (Table II), then $\alpha_{II} - \alpha_I = 9° 36' - 2° 24' = 7° 12' = \phi$ with E as origin. At the end of the first chord length from E towards I we have $x = 30'$, $y = .03$ = co-ordinates for F; $x_I = 60$, $y_I = .21$ = co-ordinates for G; $x_{II} = 90$, $y_{II} = .70$ = co-ordinates for H; $x_3 = 119.98$, $y_3 = 1.67$ = co-ordinates for I; $x_t = 60$; $y_t = .21$; $f = 42$.

The method of laying out these co-ordinates is shown in Fig. 3, in which the origin A corresponds to the point E in this example, and $A_I D_I$ becomes a curved axis with a radius of 1432.5.

If the transition curve were laid off from $B_{II} A_{II}$ as axis and B_{II} as origin, the above values x &c and y &c would be just the same except they would be laid off from the convex side of $A_{II} B_{II}$ instead of from the concave side, as was the case with $A_I B_I$ as axis. If the curvature of the circular curve is of fractional degree the value of f and the last values of x, y and α will have to be computed by the formulæ at the head of the respective column in the table.

TABLE Ia TO Va BY DEFLEXION.

§ 26. Given a tangent at any point of a transition curve as D to locate any other points as A, B, C, E, F, G. As in the case of the Tables for rectangular co-ordinates they are equally applicable to locating the points of the transition curve uniting a tangent to the circular curve or circular curves of different radii with each other. (Fig. 4)

EXAMPLE 2. Let $L_{II} - L_I = L = 120$ be the length of the transition curve; $r_I = 1910$; $r_{II} = 955$, the radii of the circular curves to be united by L by deflections from a tangent at D, where the curvature corresponds to r_I. The tangent will be common to the circle and transition curve whose rate of curvature "D" (Table II) $= 3°$. DE, EF, FG etc., being chords of the transition curve each $= 30$ feet. The tangent at D is a tangent to the circular arc with $r_I = 1910$. Then by the formula $\omega = D\Delta L + \frac{\alpha}{3}$, in which $D = 3°$; $\Delta = \frac{3}{10}$; $DE = L = 30$ feet; $\frac{\alpha}{3} = 0° 3'$; $3 \times .3' \times 30 + 0°03' = \omega = 27' + 03' = 0° 30'$ = the deflection from tangent at D to locate point E. If we want to locate the point F from D, then $L = 60$; $D = 3\Delta = 13'$; $\frac{\alpha}{3} = 12'$; $\omega = 54 + 12 = 66' = 1° 06'$ = deflection from

tangent at D to locate F with measurement from D to E thence E to F. If we wish to locate the points C, B and A from a tangent at D, then the deflection for any point C = the deflection for 30 feet for a 3 degree curve minus the deflection for the transition cnrve from A to B, or

$$\omega = D\,\Delta\,L - \frac{a}{3} = 3 \times 3 \times 30 - \frac{9}{3} = 27' - 3' = 24'.$$

The deflection for the point $B = 3 \times .3' \times 60 - \frac{36'}{3} = 0°54' - 0°12' = 0°42.$

If we have run the curve from A to D and changed the instrument to D in order to place the line of sight tangent to D, take a back sight on A and deflect 0° 54′ we have a tangent at D. To facilitate the use of the tables it is best to set the vernier at 0° 54′ and set the telescope on line AD, turn the vernier to O and continue deflection as tabulated, reading downward from D, locating the points E, F, G and C. If the curve is being run from D towards A then set the vernier at the angle indicated for any angle G, when back sight is on G from D deflect from zero and continue to deflect the angles tabulated in succession, reading up the column from D to locate C, B and A. The degree of the curvature at the instrument point controls the deflections either way. The above explanation enables us to run the transition curve from the point of greatest radii to that of its least radius, and vice versa.

To use the tables when the beginning or end of the transition curve merges into a circular curve of a fractional degree, as joining a 2½ degree curve with a 6 degree curve, or a 2 degree curve with a 5½ degree curve; compute

$$\frac{a}{3} = \frac{15}{6r} = \delta = 00'.75$$

for 15 feet, then

$$2.5 \times .3' \times 15 + 00'.75 = 0° 12',$$

which locates a point D at 3 degrees curvature; then from D, with back sight on $C + 15$;

$$2.5 \times .3' \times 15 + 01'.50 = 0° 12'.75$$

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will be the deflection to fix a tangent at D from which point proceed with deflections down the D column. If the last chord FG is one-half L, the deflections of the table can be followed up to F to which point the instrument can be moved and tangent established from which deflect

$$\omega = D\Delta \frac{L}{2} + 00'.75 = 5 \times .3' \times 15 + 00'.75 = 0^\circ\ 23'.$$

If it is desired to deflect the whole angle from D to G when FG = one-half L when L_F = the length from D to F,

$$L_G = L_F + \frac{L}{2}, \text{ then}$$

$$\alpha = \frac{L_G}{2r} \times 57^\circ\ 18'; \quad L_G = DG = 75';$$

$$r = 2296; \quad \frac{L_\alpha}{6r} \times 57^\circ\ 18' = \delta = 0^\circ\ 19';$$

$$\omega = D\Delta \left(L_F + \frac{L}{2} \right) + \frac{L_\alpha}{6r} \times 57^\circ\ 18' =$$

$$3 \times .3' \times 75 + 19' = 67' + 19' = 1^\circ\ 26'.$$

Any other fraction of L than ½ may be used in the formulæ.

TABLE I.

$P = 229200$; $i = 400$; $v = 42.4$ miles per hour.

Curvature Deg. of	$L_{,}$ or $L_{,,}$ $= \frac{P}{r}$	$\alpha^{\circ}{}_{,}$ or $\alpha^{\circ}{}_{,,}$ $= \frac{L}{2r} \times 57° 18'$	$x_{,,}$ or $x_{,}$ $= L \times \left(1 - \frac{L^2}{40r^2}\right)$	y $= \frac{L^2}{6r}$	f $= \frac{L^2}{24r} = \frac{y}{4}$	x_f $= x - r \sin \alpha$	y_f $= \frac{L^2}{48r} = \frac{f}{2}$	d $= L - \left(x_f + \frac{L}{2}\right)$	$\alpha^{\circ}{}_{,,}$ or $\alpha^{\circ}{}_{,}$ $= \frac{L}{2r} \times 57.3°$	r $= \frac{P}{L}$
A 0	00									
B 1	40	0° 12′	40.00						0 20°	5730 0
C 2	80	0° 48′	80.00	.37	.09	40.00	.05		0 80°	2865.0
D 3	120	1° 48′	119.97	1 26	.32	60.00	.16		1.80°	1910.0
E 4	160	3° 12′	159 92	2.98	.75	80 00	.37		3.20°	1432.5
F 5	200	5° 00′	199.86	5.81	1.45	100.00	.73		5.00°	1146 0
G 6	240	7° 12′	239.61	10.04	2.51	119 90	1.25		7.20°	955 0

TABLE II.

$P = 171900$; $i = 400$; $v = 37$ miles per hour.

A 0										
B 1	30	0° 09′	30.00	.03	.006	15.00	.003		0.15°	5730.0
C 2	60	0° 36′	60.00	.21	.05	30.00	.03		0.60°	2865.0
D 3	90	1° 21′	90.00	.70	.17	45.00	.09		1.35°	1910.0
E 4	120	2° 24′	119 98	1.67	.42	60.00	.21		2.40°	1432.5
F 5	150	3° 45′	149.94	3 27	.81	75.00	.41		3 75°	1146.0
G 6	180	5° 24′	179 85	5.65	1.42	89.97	.71	.03	5 40°	955.0
H 7	210	7° 21′	209.65	8.96	2.24	104 96	1 11	04	7.35°	818.8
I 8	240	9° 36′	239.33	13.38	3.35	119.93	1.67	.07	9 60°	716 3

TABLE III.

$P = 114600$; $i = 30'$; $v = 34\frac{1}{2}$ miles per hour.

Curvature Deg. of	L_1 or L_{11} $= \frac{P}{r}$	α_1 or α_{11} $= \frac{L}{2r} \times 57° 18'$	x_{11} or x_1 $= L \times \left(1 - \frac{L^2}{40r^2}\right)$	y $= \frac{L^2}{6r}$	f $= \frac{L^2}{24r} = \frac{y}{4}$	x_f $= x - r \sin \alpha$	y_f $= \frac{L^2}{48r} = \frac{f}{2}$	d $= L - \left(x_f + \frac{L}{2}\right)$	$\alpha°_{11}$ or $\alpha°_1$ $= \frac{L}{2r_1} \times 57.3°$	r $\frac{P}{L}$
A 0										
B 1	20									5730.0
C 2	40	0° 24′	40.00	.09	.02	20.00	.01		0.40°	2865.0
D 3	60	0° 54′	60.00	.31	.08	30.00	.04		0.90°	1910.0
E 4	80	1° 36′	80 00	.74	.18	40.00	.09		1.60°	1432.5
F 5	100	2° 30′	100.00	1.45	.36	50.00	.18		2.50°	1146.0
G 6	120	3° 36′	120.00	2.51	.63	60.00	.31		3.60°	955 0

TABLE IV.

$P = 85950$ $i = 30'$ $v = 30$ miles per hour.

A 0										
B 1		0° 9′								
C 2	30	0° 18′	30.00	04	.01	15.00	.005		0 30°	2865.0
D 3	45	0° 40′	45 00	.19	05	22.50	.02		0.67°	1910.0
E 4	60	1° 12′	60 00	.42	10	30.00	.05		1.20°	1432.5
F 5	75	1° 52′	75 00	.81	20	37.50	10		1.85°	1146.0
G 6	90	2° 42′	90 00	1 42	35	40.00	.18		2 70°	955.0
H 7	105	3° 45′	104.98	2.24	56	52.50	.28		3 61°	818.8
I 8	120	4° 48′	119 94	3 35	84	59 95	.42		4.80°	716.3

TABLE V.

$P = 6876'0$; $i = 3°0'$; $v = 26\frac{2}{3}$ miles per hour.

Curvature Deg. of	L or $L_{''}$ $= \frac{P}{r}$	$\alpha°$, or $\alpha°_{''}$ $= \frac{L}{2r} \times 57° 18'$	x, or $x_{''}$ $= L \times \left(1 - \frac{L^2}{40r^2}\right)$	y $= \frac{L^2}{6r}$	f $= \frac{L^2}{24r} = \frac{y}{4}$	x_f $= x - r \sin \alpha$	y_f $= \frac{L^2}{48r} = \frac{f}{2}$	d $= L - \left(x_f + \frac{L}{2}\right)$	$\alpha°_{''}$ or $\alpha°_{'}$ $= \frac{L}{2r} \times 57.3°$	r $\frac{P}{L}$
C 2	24	0° 15′	24.00	.03	.01	12.00	.004		0 25′	2865 0
D 3	36	0° 32′	36.00	.11	.03	18 00	.014		0.58°	1910.0
E 4	48	0° 58′	48.00	.27	.07	24.00	.033		0.97°	1432.5
F 5	60	1° 30′	60.00	.52	.13	30.00	.06		1.50°	1146.0
G 6	72	2° 10′	72.00	.91	.23	36.00	.11		2.17°	955.0
H 7	84	2° 56′	84.00	1.43	.36	42.00	.18		2.94°	818 8
I 8	96	3° 51′	96.00	2.15	.54	48.00	27		3.85°	716.3
J 9	108	4° 52′	107.95	3.05	.76	53.98	.38		4 87°	636.7
K 10	120	6° 00′	119.90	4.19	1.05	59.95	.52		6.00°	573.0

TABLE VI.

$P = 4584'0$; $i = 3°0'$; $v = 21.8$ miles per hour.

1	8									5730.0
2	16	0° 09′	16.00	.015	.004	8.00	.002		0.15°	2865.0
3	24	0° 22′	24.00	.05	.012	12.00	.006		0.37°	1910.0
4	32	0° 39′	32.00	.12	.03	16.00	.015		0.67°	1432.5
5	40	1° 00′	40.00	.23	.06	20.00	.03		1.00°	1146.0
6	48	1° 26′	48.00	.40	.10	24.00	.05		1.43°	955.0
7	56	1° 57′	56.00	.63	.16	28.00	.08		1.95°	818.8
8	64	2° 33′	64.00	.96	.24	32.00	.12		2.55°	716.3
9	72	3° 14′	72.00	1.36	.34	36.00	.17		3.23°	636.7
10	80	4° 00′	80.00	1.86	.47	40.00	.23		4.00°	573.0
11	88	4° 50′	88.00	2.47	.62	43.99	.31		4.83°	520.9
12	96	5° 45′	95.95	3.21	.80	48.00	.40		5.75°	477.5
13	104	6° 45′	103.90	4.08	1.02	51.99	.51	.01	6.75°	440.8
14	112	7° 50′	111.80	5.11	1.28	55.98	.64	.03	7.83°	409.3
15	120	9° 00′	119.70	6.28	1.57	59.97	.78	.05	9.00°	382.0

TABLE VII.

$P = 22920$; $i = 200$; $v = 18.9$ miles per hour.

Curvature Deg. of	L, or $L_{,,}$ $= \frac{P}{r}$	$\alpha^{\circ}{}_{,}$ or $\alpha^{\circ}{}_{,,}$ $= \frac{L}{2r} \times 57°18'$	$x_{,,}$ or $x_{,}$ $= L \times \left(1 - \frac{L^2}{40r^2}\right)$	y $= \frac{L^2}{6r}$	f $= \frac{L^2}{24r} = \frac{y}{4}$	x_f $= x - r \sin \alpha =$	y_f $= \frac{L^2}{48r} = \frac{f}{2}$	d $= L - \left(x_f + \frac{L}{2}\right)$	$\alpha^{\circ}{}_{,,}$ or $\alpha^{\circ}{}_{,}$ $= \frac{L}{2r} \times 57.3°$	r $\frac{P}{L}$
2	8									
4	16									1432.5
6	24	0° 43′	24.00	.10	.02	12.00	.01		0.72°	955.0
8	32	1° 16′	32.00	.24	.06	16.00	.03		1.27°	716 3
10	40	2° 00′	40.00	.47	12	20.00	.06		2.00°	573.0
12	48	2° 52′	48.00	.80	20	24.00	.10		2.85°	477.5
14	56	3° 55′	56.00	1 27	.32	28.00	.16		3.92°	409.3
16	64	5° 08′	63 95	1.90	.48	32.00	.24		5.13°	358.1
18	72	6° 29′	71.90	2.72	.68	36.00	39		6.48°	318 3
20	80	8° 00′	79 80	3 72	.93	40.00	.46		8.00°	286 5

TABLE VIII.

$P = 11460$; $i = 200$; $v = 13\frac{1}{2}$ miles per hour.

5	10									
10	20	1° 00′	20.00	.11	.03	10.00	.02		1.00°	573.0
15	30	2° 15′	29.99	.39	.10	15.00	.05		2.25°	382.0
20	40	4° 00′	39.98	.93	23	20.00	.12		4.00°	286.5
25	50	6° 15′	49.94	1.81	.45	24.99	.22	.01°	6.25°	229.2
30	60	9° 00′	59 85	3.14	.78	29.97	.39	.03°	9.00°	191.0

TABLE IX.

$P = 1719$; $i = 100$; $v = 7\frac{1}{6}$ miles per hour.

Curvature Deg. of	L_I or L_{II} $= \frac{P}{r}$	α°_I, or α°_{II} $= \frac{L}{2r} \times 57° 18'$	x_I, or x_{II} $= L \times \left(1 - \frac{L^2}{40r^2}\right)$	y $= \frac{L^2}{6r}$	f $= \frac{L^2}{24r} = \frac{y}{4}$	x_f $= x - r \sin \alpha$	y_f $= \frac{L^2}{48r} = \frac{f}{2}$	d $= L - \left(x_f + \frac{L}{2}\right)$	α°_{II} or α°_I $= \frac{L}{2r} \times 57.3°$	r $\frac{P}{L}$
10	3	0° 09′	3.00	.003					0.15°	573.0
20	6	0° 36′	6.00	.02	.005				0.60°	286.5
30	9	1° 21′	9.00	.07	.02		.01		1.35°	191.0
40	12	2° 24′	12.00	.17	04		.02		2.40°	143.2
50	15	3° 45′	15.00	.33	.08	7.50	.04		3.75°	114.6
60	18	5° 24′	17.99	.56	.14	9.00	.07		5.40°	95.5
70	21	7° 20′	20.96	.90	.22	10.50	.11		7.33°	81.8
80	24	9° 35′	23 93	1.34	.33	12.00	.17		9.58°	71.7
90	27	12° 08′	26.88	1.91	.48	13.49	.24	.01	12.28°	63.7
100	30	15° 00′	29.79	2.62	.65	14.96	.33	.04	15.00°	57.3
110	33	18° 09′	32 66	3 48	87	16.43	.44	.07	18.15°	52.1
120	36	21° 40′	35.48	4.52	1.13	17.90	.56	.10	21.67°	47.7

Δ = deflection angle per foot for 1° circular curve.

$\delta = \frac{a}{3}$ deflection angle from tangent at A towards G for points of spiral.

D = degree or rate of curvature at position of instrument.

ω = deflection from tangent at any point of spiral to any other point of spiral. Then

$$\omega = D\,\Delta\,L \pm \frac{a}{3} = D\,\Delta\,L \pm \frac{L}{6\,r}\,57°\,18'$$

+ sign for running towards G — sign towards A.

The instrument point is the origin of L and x. x conceived to to be laid off on the circular curve passing through instrument point as axis

TABLE Ia BY DEFLECTIONS

$P = 229200$; $i = 400$; $v = 42.4$ per hour.

L	x	D	$D\,\Delta$	$\delta = \frac{a}{3}$	$\omega = D\,\Delta\,L - \frac{L}{6\,r}\,57°\,18'$						$\delta = \frac{a}{3}$
0	0	A 0°	00	Tan at A	0° 08′	0° 32′	1° 12′	2° 08′	3° 20′		4° 48′
40	40.0	B 1°	°00.3′	0° 04′	Tan at B	0° 20′	0° 56′	1° 48′	2° 56′		4° 20′
80	80.0	C 2°	°00.6′	0° 16′	0° 16′	Tan at C	0° 32′	1° 20′	2° 24′		3° 44′
120	120 0	D 3°	°00.9′	0° 36′	0° 40′	0° 28′	Tan at D	0° 44′	1° 44′		3° 00′
160	159.9	E 4°	°01.2′	1° 04′	1° 12′	1° 04′	0° 40′	Tan at E	0° 56′		2° 08′
200	199.8	F 5°	°01.5′	1° 40′	1° 52′	1° 48′	1° 28′	0° 52′	Tan at F		1° 08′
240	239.6	G 6°	°01.8′	2° 24′	2° 40′	2° 40′	2° 24′	1° 52′	1° 04′		Tan at G

$$\omega = D\,\Delta\,L + \frac{L}{6\,r}\,57°\,18'$$

TABLE IIa BY DEFLECTIONS.

$P = 171900$; $i = 400$; $v = 37$ miles per hour

L	x	D	$D\triangle$	$\delta = \frac{a}{3}$	$\omega = D\triangle L - \frac{L}{6r}57°18'$							$\delta = \frac{2}{3}a$
00	00	A 0°		Tan at *A*	0° 06′	0° 24′	0° 54′	1° 36′	2° 30′	3° 36′	4° 54′	6° 24′
30	30	B 1°	°00.3′	0° 03′	Tan at *B*	0° 15′	0° 42′	1° 21′	2° 12′	3° 15′	4° 30′	5° 57′
60	60	C 2°	°00.6′	0° 12′	0° 12′	Tan at *C*	0° 24′	1° 00′	1° 48′	2° 48′	4° 00′	5° 24′
90	90	D 3°	°00.9′	0° 27′	0° 30′	0° 21′	Tan at *D*	0° 33′	1° 18′	2° 15′	3° 24′	4° 45′
120	120	E 4°	°01.2′	0° 48′	0° 54′	0° 48′	0° 30′	Tan at *E*	0° 42′	1° 36′	2° 42′	4° 00′
150	149.9	F 5°	°01.5′	1° 15′	1° 24′	1° 21′	1° 06′	0° 39′	Tan at *F*	0° 51′	1° 54′	3° 09′
180	179.8	G 6°	°01.8′	1° 48′	2° 00′	2° 00′	1° 48′	1° 24′	0° 48′	Tan at *G*	1° 00′	2° 12′
210	209.6	H 7°	°02.1′	2° 27′	2° 42′	2° 45′	2° 36′	2° 15′	1° 42′	0° 57′	Tan at *H*	1° 09′
240	239.3	I 8°	°02.4′	3° 12′	3° 30′	3° 36′	3° 30′	3° 12′	2° 42′	2° 00′	1° 06′	Tan at *I*

$\omega = D\triangle L + \frac{L}{6r}57°18'$

TABLE IIIa BY DEFLECTIONS

$P = 114600$; $i = 300$; $v = 31\frac{1}{2}$ miles per hour.

L	x	D	$D\Delta$	$\delta = \frac{\omega}{3}$	$\omega = D\Delta L - \frac{L}{6r}57° 18''$					$\delta = \frac{2}{3}\omega$
				Tan at A						
0	00	A 0°	00		0° 04′	0° 16′	0° 36′	1° 04′	1° 40′	2° 24′
20	20	B 1°	°00.3′	0° 02′	Tan at B	0° 10′	0° 28′	0° 54′	1° 28′	2° 10′
40	40	C 2°	°00.6′	0° 08′	0° 08′	Tan at C	0° 16′	0° 40′	1° 12′	1° 52′
60	60	D 3°	°00.9′	0° 18′	0° 20′	0° 14′	Tan at D	0° 22′	0° 52′	1° 30′
80	80	E 4°	°01.2′	0° 32′	0° 36′	0° 32′	0° 20′	Tan at E	0° 28′	1° 04′
100	100	F 5°	°01.5′	0° 50′	0° 56′	0° 54′	0° 44′	0° 26′	Tan at F	0° 34′
120	120	G 6°	°01.8′	1° 12′	1° 20′	1° 20′	1° 12′	0° 56′	0° 32′	Tan at G
				$\omega = D\Delta L + \frac{L}{6r}57° 18'$						

TABLE IVa BY DEFLECTIONS.

$P = 8595$; $i = 300$; $v = 30$ miles per hour

L	x	D	$D\triangle$	$\delta = \frac{a}{3}$	$\omega = D\triangle L - \frac{L}{6r}57°\,18'$							$\delta = \frac{2}{3}a$
00	00	A 0°	°00.0′	Tan at *A*	0° 03′	0° 12′	0° 27′	0° 48′	1° 15′	1° 48′	2° 26′	3° 12′
15	15	B 1°	°00.3′	0°01.5′	Tan at *B*	0° 7′	0° 21′	0° 41′	1° 06′	1° 38′	2° 15′	2° 59′
30	30	C 2°	°00.6′	0° 06′	0° 06′	Tan at *C*	0° 12′	0° 30′	0° 54′	1° 24′	2° 00′	2° 42′
45	45	D 3°	°00.9′	0° 13′	0° 15′	0° 11′	Tan at *D*	0° 16′	0° 39′	1° 08′	1° 42′	2° 23′
60	60	E 4°	°01.2′	0° 24′	0° 27′	0° 21′	0° 15′	Tan at *E*	0° 21′	0° 48′	1° 21′	2° 00′
75	75	F 5°	°01.5′	0° 37′	0° 42′	0° 40′	0° 33′	0° 20′	Tan at *F*	0° 25′	0° 57′	1° 35′
90	90	G 6°	°01.8′	0° 54′	1° 00′	1° 00′	0° 54′	0° 42′	0° 24′	Tan at *G*	0° 30′	1° 06′
105	105	H 7°	°02.1′	1° 13′	1° 21′	1° 21′	1° 18′	1° 07′	0° 51′	0° 29′	Tan at *H*	0° 33′
120	119.9	I 8°	°02.4′	1° 36′	1° 45′	1° 48′	1° 45′	1° 36′	1° 21′	1° 00′	0° 33′	Tan at *I*

$\omega = D\triangle L + \frac{L}{6r}57°\,18'$

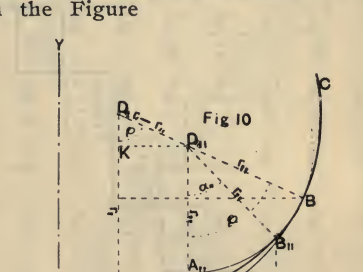
TABLE Va BY DEFLECTIONS

$P = 68760$; $i = 300$; $v = 26\frac{2}{3}$ miles per hour

L	x	D	$D\triangle$	$\delta = -\frac{a}{3}$	$\omega = D\triangle L - \frac{L}{6r}\,57°\,18'$								$\delta = \frac{2}{3}a$
0	0	A 0°	°00.0′	Tan at *A*	02½′	09′	21′	39′	1° 00′	1° 28′	1° 57′	2° 33′	3° 15′	4° 00′
12	12	B 1°	°00.3′	0° 1¼′	Tan at *B*	06′	16′	32′	53′	1° 18′	1° 48′	2° 22′	3° 02′	3° 47′
24	24	C 2°	°00.6′	0° 05′	05′	Tan at *C*	10′	24′	43′	1° 07′	1° 36′	2° 10′	2° 48′	3° 31′
36	36	D 3°	°00.9′	0° 11′	12′	08′	Tan at *D*	13′	31′	54′	1° 22′	1° 54′	2° 31′	3° 13′
48	48	E 4°	°01.2′	0° 19′	22′	19′	12′	Tan at *E*	17′	38′	1° 05′	1° 36′	2° 12′	2° 53′
60	60	F 5°	°01.5′	0° 30′	33′	33′	27′	15′	Tan at *F*	21′	45′	1° 15′	1° 51′	2° 30′
72	72	G 6°	°01.8′	0° 43′	48′	48′	43′	34′	19′	Tan at *G*	24′	53′	1° 26′	2° 05′
84	84	H 7°	°02.1′	0° 59′	1° 05′	1° 06′	1° 02′	54′	41′	22′	Tan at *H*	28′	1° 00′	1° 37′
96	96	I 8°	°02.4′	1° 17′	1° 24′	1° 26′	1° 24′	1° 17′	1° 05′	48′	26′	Tan at *I*	32′	1° 07′
108	108	J 9°	°02.7′	1° 37′	1° 47′	1° 49′	1° 48′	1° 42′	1° 31′	1° 16′	0° 55′	30′	Tan at *J*	35′
120	119.9	K 10°	°03.0′	2°00′	2° 09′	2° 15′	2° 15′	2° 09′	2° 00′	1° 45′	1° 26′	1° 03′	33′	Tan at *K*

$\omega = D\triangle L + \frac{L}{6r}\,57°\,18'$

To introduce the transition curve in alignment where simple curves have been run. In Fig 10, suppose ABC, a simple circular curve, to have been run tangent to the line $OG_{,,}$ at A with radius $r_{,}=D_{,}A=D_{,}B$; it is desired to introduce a spiral whose greatest curvature has a radius $r_{,,}=D_{,,}A_{,,}$, $D_{,,}B_{,,}$, $D_{,,}B<r_{,}$. From the tables or by computation we have m depending on the value of $r_{,,}$ and $L_{,,}$; with given values of $m, r_{,,}$ and $r_{,}$ we have from the Figure



$$A\,D_{,} = A_{,}A_{,,} + A_{,,}D_{,,} + D_{,,}D_{,}\cos D_{,,}D_{,}K;$$

or
$$r_{,} = m_{,} + r_{,,} + (r_{,} - r_{,,})\cos\phi\,.\,.$$

$$m = r_{,} - r_{,,} - (r_{,} - r_{,,})\cos\phi$$

$$m = r_{,} - r_{,,}(1 - \cos\phi) = (r_{,} - r_{,,})\,\mathrm{ver}\,\phi$$

$$\mathrm{ver}\,\phi = \frac{m}{r_{,} - r_{,,}};\quad r_{,}\phi = AB;\quad r_{,,}\phi = A_{,,}B$$

AB = the distance to measure from PC to locate B; $BA_{,,}$ the distance to measure from B on $BA_{,,}$ to locate $A_{,,}$. From m and $r_{,,}$ we have

$$2\,r_{,,}\alpha = L = \sqrt{24\,m\,r_{,,}},\ \text{squaring},\ 4\,r_{,}^{2}\,\alpha_{,,}^{2} = 24\,m\,r_{,,}$$

or
$$\alpha_{,,}^{2} = \frac{6\,m}{r_{,,}};\quad \alpha_{,,} = \sqrt{\frac{6\,m}{r_{,,}}};\quad r_{,,}\alpha_{,,} = A_{,,}B_{,,} = \frac{L}{2}$$

$$r_{,,}(\phi_{,,} - \alpha_{,,}) = BB_{,,};\quad (r - r_{,})\sin\phi = A_{,}A$$

The rate of curvature of $r_{,,}$ should not be more than from 1° to 2° greater than that of $r_{,}$ (when possible) for curves of a curvature less than 10°; 2° to 3° difference, for 10° to 15° rate of curvature; 3° to 5° difference for 15° to 20° rate of curvature.

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